M.SC. MATHEMATICS

MAL-641

Functional Analysis

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LIST OF SYMBOLS

$\forall$	For all
$\in$	Element of
$\exists$	There exist
$\subset$	Proper subset of
$\subseteq$	Subset of
$\cup$	Union
$\cap$	Intersection
$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of positive integer
$\mathbb{R}^n$	n-dimentional euclidean space
$\sum$	Sigma
$\prod$	Summation
I_x	Identity mapping
Sup	Supremum
Inf	Infimum
Max	Maximum
Min	Minimum
opt	Supremum or infimum
N	Normed Linear Spaces
B	Banach Spaces
H	Hilbert Spaces
$[a,b]$	Closed interval
(a,b)	Open interval
(X,d)	Metric space
$(X,\preceq)$	Partalliy ordered set

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CHAPTER 1

Normed Linear Space and Inequalities

In this Chapter, we shall discuss some preliminaries on inequalities, vector spaces and metric spaces, which can be usefull as a reference material. Further, a study of normed linear space and its propeties is made. Some examples on space are also discussed. Finally, the chapter concludes with Cauchy, Holders and Minkowski's Inequalities.

Definition 1.1. A vector space or linear space over field F is a set X with operation called addition $X \times X$ to X given by $(x,y) \to x+y$ and an operation called scalar multiplication defined on $F \times X \to X$ given by $(\alpha, x) \to \alpha x$ satisfying the following conditions. For all $x,y,z \in X$ and $\alpha, \beta \in F$

(i) $(x+y)+z = x+(y+z)$

(ii) $x+y = y+x$

(iii) $\exists$ an element $0 \in X$ such that $x+0 = x = x+0$

(iv) for each $x \in X$ $\exists$ an element $-x \in X$ such that $x+(-x) = 0 = (-x)+x$

(v) $\alpha(x+y) = \alpha x + \alpha y$

(vi) $(\alpha+\beta)x = \alpha x + \beta x$,

(vii) $(\alpha\beta)x = \alpha(\beta x)$

(viii) $1.x = x$

The two primary operations in a linear space addition and scalar multiplication are called the linear operations. The zero element of a linear space is usually referred to as the origin.

A linear space is called a real linear space or a complex linear space according as the scalars are real numbers or complex numbers.

Examples of Vector Space

Example 1.2. The set R of all real numbers is a real linear space under addition and multiplications of real numbers. R is not a vector space over C.

Example 1.3. The set C of all complex numbers is a complex linear space under addition and multiplications of complex numbers.

Example 1.4. For any positive interger n,

$$R^n = \{(x_1, x_2, \cdots, x_n) : x_i \in R, i = 1, 2, \cdots n\}$$

is vector space over with operations

$$x + y = (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

and

$$\alpha x = \alpha(x_1, x_2, \cdots, x_n) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n).$$

The above operations are called coordinate-wise operations. In a similar manner, C^n is linear space over field of complex numbers.

Linear Transformation

Let U and V be two vector space over same field F. A mapping $T : U \to V$ is said to be linear if

(i) $T(x + y) = T(x) + T(y)$ for all $x, y \in X$

(ii) $T(\alpha x) = \alpha T(x)$ for every $x \in X$ and every $\alpha \in F$.

Definition 1.5. An isomorphism f between linear spaces (over the same scalar field) is a bijective linear map that is f is bijective and

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)b$$

Two linear spaces are called isomorphic (or linearly isomorphic) if and only if there exists an isomorphism between them.

The notion of norm was established in order to give a method to measure the lengh (magnitude) of a vector. For example, if $x = (-1, 2, -3, -7, -11)$ is in R^5, then $||x|| = 11$ is vector norm, which is length of largest coordinate. On the real line norm of vector $||x|| = |x|$. In fact the concept of norm is generalization of concept of length that is familiar for the set of real or complex numbers.

Definition 1.6. A semi-norm on a linear space X is a function $\rho : X \to R$ satisfying

(i) $\rho(x) \geq 0 \; \forall \, x \in X$.

(ii) $\rho(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and α (scalar)

(iii) $\rho(x + y) \leq \rho(x) + p(y)$ for all $x, y \in X$.

Property (ii) is called absolute homogeneity of ρ and property (iii) is called subadditivity of ρ. Thus a semi-norm is non-negative real, subadditive, absolutely homogeneous function of the linear space e.g. $\rho(x) = |x|$ is a seminorm on the linear space C of complex numbers. Similarly if $f : X \to C$ is a linear map, then $\rho(x) = |f(x)|$ is a semi-norm on x.

Thus a semi-normed linear space is an ordered pair (x,ρ) where ρ is a seminorm on x.

Definition 1.7. Let X be real (complex) linear space then a norm on a linear space X is a function $\|\ \| : X \to R$ satisfying

(i) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$ for $x \in X$

(ii) $\|\alpha x\| = |\alpha|.\|x\|$

(iii) $\|x+y\| \leq \|x\| + \|y\|$

we observe that a semi-norm becomes a norm if it satisfies one additional condition i.e.

$$\|x\| = 0 \quad \text{iff } x = 0$$

Further, $\|x\|$ is called norm of x. The non-negative real number$\|x\|$ is considered as the length of the vector x.

A normed linear space is an ordered pair $(X,\|.\|)$ where $\|.\|$ is a norm on X.

Metric on Normed linear Spaces

Definition 1.8. Let N be an arbitrary set. It is called a metric space if there exists a function $d : N \times N \to R$ (called distance or metric function) satisfying

(i) $d(x,y) \geq 0$

(ii) $d(x,y) = 0$ if and only if $x = y$.

(iii) $d(x,y) = d(y,x)$

(iv) $d(x,z) \leq d(x,y) + d(y,z)$ [Triangle inequality] for any $x,y,z \in N$.

If d is metric on X, then the ordered pain (N,d) is called a metric space.

Let N be a normed linear space. We introduce a metric in N defined by

$$d(x,y) = \|x-y\| \tag{1}$$

This metric (distance function) satisfies all axioms of the definition of norm. As

(i) $\|.\| \geq 0 \Leftrightarrow d(x,y) \geq 0$.

(ii) $d(x,y) = 0 \Leftrightarrow \|x-y\| = 0 \Leftrightarrow x-y = 0 \Leftrightarrow x = y$.

(iii) $d(x,y) = \|x-y\| = \|-1(y-x)\| = |-1|\|y-x\| = \|y-x\| = d(y,x)$

(iv) $d(x,y) = \|x-y\| = \|x-y+z-z\| \leq \|x-z\| + \|z-y\| = d(x,z) + d(z,y)$.

Hence a normed linear space N is a metric space with respect to the metric d defined above. But every metric space need not be a normed linear space since in every metric space there need not be a vector space structure defined e.g. the vector space $X \neq 0$ with the discrete metric defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

is not a normed linear space.

Note: The above metric in equation 1, has following additional properties: If $x,y,z \in N$ and α a scalar, then

(i) $d(x+z, y+z) = \|(x+z)-(y+z)\| = \|x-y\| = d(x,y)$ (Translation Invariance)

(ii) $d(\alpha x, \alpha y) = \|\alpha x - \alpha y\| = \|\alpha(x-y)\| = |\alpha|\|x-y\| = |\alpha| d(x,y)$.

Also it is important to note here that a metric is induced norm only when the above two properties are satisfied for all value of scalar α.
For discrete metric, If we set $\alpha = 2$, then

$$d(\alpha x, \alpha y) \neq |\alpha| d(x,y)$$

as

$$d(2x, 2y) = \begin{cases} 0 & \text{if } 2x = 2y \text{ i.e., } x = y \\ 1 & \text{if } 2x \neq 2y \text{ i.e., } x \neq y \end{cases}$$

and

$$2d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2 & \text{if } x \neq y. \end{cases}$$

Remark: In the definition of norm $\|x\| = 0 \Leftrightarrow x = 0$ is equivalent to the condition

$$\|x\| \neq 0 \;\text{ if } x \neq 0$$

Also the fact that $\|x\| > 0$ is implied by the second and third condition of norm

$$\|0\| = \|0.1\| = 0.\|1\| = 0$$

and $\|0\| = \|x - x\| \leq \|x\| + \|x\| = 2\|x\|$

$$\Rightarrow \; 2\|x\| \geq 0$$

$$\Rightarrow \; \|x\| \geq 0.$$

Remark: As in the case of real line, the continuity of a function can be given in terms of convergence of certain sequence. We can alternatively define continuity in terms of convergence of sequence in normed linear space also.

Definition 1.9. Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two normed linear spaces respectively. We say that f is continuous at $x_0 \in E$ if given $\in > 0, \exists\, \delta > 0$ such that

$$\Rightarrow \qquad \|f(x) - f(x_0)\|_F < \in \text{ whenever } \|x - x_0\|_E < \delta.$$

Since every normed linear space is a metric space, this definition of continuity is same in it as the definition of continuity in metric space. Thus f is continuous at $x_0 \in E$ iff whenever $x_n \to x_0$ in E, $f(x_n) \to f(x_0)$ in F.

Remark: In normed linear spaces, convergence is defined as

$$x = \lim_n x_n \text{ or } x_n \to x \text{ by } \|x_n - x\| \to 0 \text{ as } n \to \infty$$

This convergence in normed linear space is called **convergence in norm** or **strong convergence.**

Definition 1.10. A sequence $< x_n >$ in a normed linear space is a Cauchy sequence if given $\in > 0$, there exists a positive integer m_0 such that

$$\|x_m - x_n\| < \in \text{ whenever } m, n \geq m_0.$$

Definition 1.11. A normed linear space N is called complete or Banach space iff every Cauchy sequence in it is convergent that is if for each Cauchy sequence $< x_n >$ in N, there exist an element x_0 in N such that $x_n \to x_0$. A complete normed linear space is called a Banach space.

Some properties of Normed Linear Spaces

Theorem 1.12. Let N be a normed linear space over the scalar field F. Then

(i) The map $(\alpha x) \to \alpha x$ from $F \times N \to N$ is continuous

(ii) The map $(x,y) \to x+y$ from $N \times N \to N$ is continuous.

(iii) The map $x \to \|x\|$ from N to R is continuous.

Proof: To prove (i) we must show that if $\alpha_n \to \alpha$ and $x_n \to x$, then $\alpha_n x_n \to \alpha x$.
So we assume $\alpha_n \to \alpha$ and $x_n \to x$ i.e. $|\alpha_n - \alpha| \to 0, \|x_n - x\| \to 0$.
Then $\|\alpha_n x_n - \alpha x\| = \|\alpha_n(x_n - x) + (\alpha_n - \alpha)x\|$

$$\leq |\alpha_n| \; \|x_n - x\| + |\alpha_n - \alpha| \,.\, \|x_n\| \to 0$$

and so (i) holds.

To prove (ii) we suppose that $x_n \to x, y_n \to y$ i.e. $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$. Then by triangle inequality

$$\begin{aligned} \|(x_n + y_n) - (x+y)\| &= \|(x_n - x) + (y_n - y)\| \\ &\leq \|x_n - x\| + \|y_n - y\| \to 0 \end{aligned}$$

and so $x_n + y_n \to x + y$ and hence (ii) holds. Before proving (iii), we establish the inequality

$$|\,\|x\| - \|y\|\,| \leq \|x - y\| \tag{*}$$

We note that in a normed linear space

$$\begin{aligned} & \|x\| = \|y + (x-y)\| \leq \|y\| + \|x - y\| \\ \Rightarrow \quad & \|x\| - \|y\| \leq \|x - y\| \end{aligned} \tag{2}$$

On interchanging the roles of x and y, we find that

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\| \tag{3}$$

From (1) and (2), it follows that

$$|\,\|x\| - \|y\|\,| \leq \|x - y\|$$

We now prove (iii). Let $x_n \to x$, then from the above inequality,

$$|\,\|x_n\| - \|x\|\,| \leq \|x_x - x\| \to 0$$

which implies that $\|x_n\| \to \|x\|$ Thus we have shown that $x_n \to x \Rightarrow \|x_n\| \to \|x\|$.

Thus the map $\|.\| : N \to R$ is continuous. Hence the result.

Remark: (i) and (ii) show that scalar multiplication and addition are jointly continuous where as (iii) shows that norm is a continuous function. (2) The introduction of a norm in a linear space is called norming.

Theorem 1.13. In a normed linear space, every convergent sequence is a Cauchy sequence.

Proof: Suppose that the sequence $< x_n >$ in a normed linear space N converges to a point $x_0 \in N$. To show that it is Cauchy sequence, let $\in > 0$ be given. Since the sequence $< x_n >$ converges to x_0, there exists a positive integer m_0 such that $n \geq m_0 \Rightarrow \|x_n - x_0\| < \frac{\in}{2}$. Hence for all $m, n \geq m_0$, we have

$$\|x_m - x_n\| = \|x_m - x_0 + x_0 - x_n\| \leq \|x_m - x_0\| + \|x_n - x_0\| < \frac{\in}{2} + \frac{\in}{2} = \in .$$

Thus the convergent sequence $< x_n >$ is a Cauchy sequence.

Further Properties of Normed spaces

By definition, a subspace Y of a normed space X is a subspace of X considered as a vector space, with the norm obtained by restricting the norm on X to the subset Y. This norm on Y is said to be induced by the norm on$X$. If $Y$is closed in X, then Y is called a closed subspace of X. Thus, a subspace Y of a Banach X is considered as a normed space. Hence we donot require Y to be complete.

Theorem 1.14. A subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

Proof: The result directly follows from “A subspace M of a complete metric space X is itself complete if and only if the set M is closed in X.

Definition 1.15. Infinite series can now be defined in a way similar to that in calculus. In fact, if $< x_k >$is a sequence in a normed space X, we can associate with $< x_k >$ the sequence $< S_n >$ of partial sums

$$S_n = x_1 + x_2 + \cdots + x_n$$

For $n = 1, 2 \ldots$. If $< S_n >$ is convergent, say $S_n \to S$ that is $\|S_n - S\| \to 0$, Then the infinite series or briefly the series

$$\sum_{K=1}^{\infty} x_K = x_1 + x_2 + \cdots \tag{1}$$

is said to converge or to be convergent, S is called the sum of the series and we write

$$S = \sum_{K=1}^{\infty} x_K = x_1 + x_2 + \cdots$$

If $S = \sum_{K=1}^{\infty} \|x_K\| = \|x_1\| + \|x_2\| + \ldots\ldots..$ converges, then the series 1 is said to be absolutely convergent. However in a normed space X absolute convergence implies convergence if and only if X is complete.

The concept of convergence of a series can be used to define a basis as follows:

If a normed space x contains a sequence $< e_n >$ with the property that for every $x \in X$, there is a unique sequence of scalars $< \alpha_n >$ such that

$$\|x - (\alpha_1 e_1 + \ldots + \alpha_n e_n)\| \to 0 \text{ as } n \to \infty \tag{6}$$

then $< e_n >$ is called a Schauder Basis for X. The series

$$\sum_{K=1}^{\infty} \alpha_K e_K$$

which has the sum x is then called the expansion of x with respect to $< e_n >$ and we write

$$x = \sum_{K=1}^{\infty} \alpha_K e_K$$

Finite Dimensional Normed Spaces and Subspaces

Theorem 1.16. Every finite dimensional subspace Y of a normed space X is complete. In particular, every finite dimensional normed space is complete.

Proof: To prove the theorem,first we prove a Lemma,

Lemma. Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space X (of any dimension). Then there is a number $C > 0$ such that for every choice of scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$, we have

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq C(|\alpha_1| + \ldots |\alpha_n|) \qquad (C > 0) \tag{1}$$

Proof of Lemma: We write $S = |\alpha_1| + |\alpha_2| + |\alpha_n|$. If $S = 0$, all α_i are zero, so that (1) holds for any C. Let $S > 0$, then (1) is equivalent to the inequality which we obtain from (1) by dividing by S and writing $\beta_j = \alpha_j / S$ that is

$$\|\beta_1 x_1 + \ldots + \beta_n x_n\| \geq C(\sum_{j=1}^{n} |\beta_j| = 1) \tag{2}$$

Hence it is sufficient to prove the existence of a $C > 0$ such that (2) holds for every n-tuple of scalars $\beta_1 \ldots \beta_n$ with

$$\sum |\beta_j| = 1.$$

Suppose that this is false. Then there exists a sequence $< y_m >$ of vectors

$$y_m = \beta_1^{(m)} x_1 + \ldots + \beta_n^{(m)} x_n \qquad (\sum_{j=1}^{n} |\beta_j^{(m)}| = 1)$$

such that

$$\|y_m\| \to 0 \quad \text{as} \quad m \to \infty.$$

Since $\sum |\beta_j^{(m)}| = 1$, we have $|\beta_j^{(m)}| \leq 1$. Hence for each fixed j, the sequence

$$< \beta_j^{(m)} > = < \beta_j^{(1)}, \beta_j^{(2)}, \ldots >$$

is bounded. Consequently, by the Bolzano-Weierstrass theorem, $< \beta_j^{(m)} >$ has a convergent subsequence. Let β_1 denote the limit of that subsequence and let $< y_{1,m} >$ denote the corresponding subsequence of $< y_{2,m} >$. By the same argument, $< y_{1,m} >$ has a subsequence $< y_2, m >$ for which the corresponding subsequence of scalars $\beta_2^{(m)}$ converges, let β_2 denote the limit-continuing in this way, after n steps we obtain a subsequence

$$< y_{n,m} > = (y_{n,1}, y_{n,2}, \ldots) \quad \text{of} \quad < y_m >$$

whose terms are of the form

$$y_{n,m} = \sum_{j=1}^{n} \gamma_j^{(m)} x_j (\sum_{j=1}^{n} \gamma_j^{(m)} | = 1)$$

with scalars $\gamma_j^{(m)}$ satisfying $\gamma_j^{(m)} \to \beta_j$ as $m \to \infty$.

Hence as $m \to \infty$,

$$y_{n,m} \to \sum_{j=1}^{n} \beta_j x_j$$

where $\sum |\beta_j| = 1$ so that not all β_j can be zero. Since $\{x_1, \ldots, x_n\}$ is a linearly independent set, we thus have $y \neq 0$. On the other hand , $y_{n,m} \to y$ implies $\|y_{n,m}\| \to \|y\|$ by the continuity of the norm. Since $\|y_m\| \to 0$ by assumption and $< y_{n,m} >$ is a subsequence of $< y_m >$, we must have $\|y_{n,m}\| \to 0$, Hence $\|y\| = 0$, so that $y = 0$. But this contradicts that $y \neq 0$, and the lemma is proved.

Now we prove the theorem.

Proof of the theorem. We consider an arbitrary Cauchy sequence $< y_m >$ in Y and show that it is convergent in Y, the limit will be denoted by y. Let $\dim Y = n$ and $\{e_1, e_2, \ldots e_n\}$ any basis for Y. Then each y_m has a unique representation of the form

$$y_m = \alpha_1^{(m)} e_1 + \ldots\ldots\ldots + \alpha_n^{(m)} e_n$$

Since $< y_m >$ is a Cauchy sequence, for every $\in > 0$, there is an N such that $\|y_m - y_n\| < \in$ when $m, r > N$. From this and the above Lemma, we have for some $C > 0$,

$$\begin{aligned} \in > \|y_m - y_r\| &= \| \sum_{j=1}^{r} (\alpha_j^{(m)} - \alpha_j^{r}) e_j \| \\ &\geq C \sum_{j=1}^{r} |\alpha_j^{(m)} - \alpha_j^{(r)}| \end{aligned}$$

where $m, r > N$. Division by $C > 0$ gives

$$|\alpha_j^{(m)} - \alpha_j^{(r)}| \leq \sum_{j=1}^{r} |\alpha_j^{(m)} - \alpha_j^{(r)}| < \frac{\in}{C}, \quad (m, r > N)$$

This shows that each of the n sequences

$$< \alpha_j^{(m)} > = < \alpha_j^{1}, \alpha_j^{(2)}, \ldots > \quad j = 2, 2, \ldots, n$$

is Cauchy in R or C. Hence it converges let α_j denote the limit. Using these n limits, $\alpha_1, \alpha_2, \ldots, \alpha_n$, we define

$$y = \alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n \alpha_n$$

Clearly $y \in Y$. Further

$$\|y_m - y\| = \|\sum_{j=1}^{n} (\alpha_j^{(m)} - \alpha_j)e_j\| \leq \sum_{j=1}^{n} |\alpha_j^{(m)} - \alpha_j|.\|e_j\|$$

On the right $\alpha_1^m \to \alpha_j$. Hence $\|y_m - y\| \to 0$, that is $y_m \to y$. This shows that $< y_m >$ is convergent in Y. Since $< y_m >$ was an arbitrary Cauchy sequence in Y.

This proves that Y is complete.

Remark. From the above theorem and the result "A subspace M of a complete metric space X is complete if and only if the set M is closed in X", we get the following:

Theorem 1.17. *Every finite dimensional subspace Y of a normed space X is closed in X.*

Remark. Infinite dimensional subspaces need not be closed e.g. Let $X = C[0,1]$ and $Y = \text{span}\{x_0, x_1, \ldots..\}$ where $x_j(t) = t^j$ so Y that is the set of polynomials. Y is not closed in X.

Quotient Space

Definition 1.18. Let M be a subspace of a linear space L and let the coset of an element x in Lbe defined by

$$x + M = \{x + m; m \in M\}$$

Then the distinct cosets form a partition of L and if addition and scalar multiplication are defined by

$$(x + M) + (y + M) = (x + y) + M$$

and

$$\alpha(x + M) \equiv \alpha x + M$$

then these cosets constitute a linear space denoted by L/M and called the quotient space of L with respect to M. The origin in L/M is the coset $0 + M = M$ and the negative of $x + M$ is $(-x) + M$.

Theorem 1.19. Let M be a closed linear subspace of a normed linear space N. If the norm of a coset $x + M$ in the quotient space N/M is defined by

$$\|x + M\| = \inf\{\|x + m\|; m \in M\} \tag{1}$$

Then N/M is a normed linear space. Further if N is a Banach space. Then so is N/M.

Proof: We first verify that (1) defines a norm in the required sense. It is obvious that $\|x+M\| \geq 0$. Since $\|x+m\|$ is a non-negative real number and every set of non-negative real numbers is bounded below, it follows that inf $\{\|x+m\|; m \in M\}$ is non negative. That is

$$\|x+M\| \leq 0 \quad \forall\, x+M \in N/M$$

Also $\|x+M\| = 0 \Leftrightarrow$ there exists a sequence $\{m_k\}$ in M such that $\|x+m_k\| \to 0$
$\Leftrightarrow$ x is in M
$\Leftrightarrow$ $x+M = M =$ The zero element of N/M.

Next we have

$$\begin{aligned}
\|(x+M)+(y+M) &= \|(x+y)+M\| \\
&= \inf\{\|x+y+m\|; m \in M\} \\
&= \inf\{\|x+y+m+m'\|; m\,\text{and}\ \in M\} \\
&= \inf\{\|(x+m)+(y+m'\|; m, m' \in M\} \\
&\leq \inf\{\|x+y+m\|; m\,\text{and}\ \in M\} + \inf\{\|y+m'\|.m' \in M\} \\
&= \|x+M\| + \|y+M\| \\
\|\alpha(x+M)\| &= \inf\{\|\alpha(x+M)\|; m \in M\} \\
&= \inf\{|\alpha|\ \|x+m\|; m \in M\} \\
&= |\alpha| \inf\{\|x+m\|; m \in M\} \\
&= |\alpha|\ \|x+M\|
\end{aligned}$$

Finally we assume that N is complete and we show that N/M is also complete.

If we start with a Cauchy sequence in N/M, Then it is sufficient to show that this sequence has a convergent subsequence. It is clearly possible to find a subsequence $\{x_n+M\}$ of the original Cauchy sequence such that

$$\|(x_2+M)-(x_2+M)\| < \frac{1}{4}$$

and in general

$$\|(x_n+M)-(x_{n+1}+M)\| < \frac{1}{2^n}$$

we prove that this sequence is convergent in N/M. We begin by choosing any vector y_1 in x_1+M and we select y_2 in x_2+M such that $\|y_1-y_2\| < \frac{1}{2}$. We next select a vectory_3 in x_1+M such that $\|y_2-y_3\| < \frac{1}{4}$. Continuing in this way we obtain a sequence $\{y_n\}$ in N such that If $m < n$, then

$$\|y_m - y_n\| = \|y_m - y_{m+1}) + (y_{m+1}y_{m+2}) + \ldots + (y_{n-1} - y_n\|$$

$$\begin{aligned}
&\le \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \ldots + \|y_{n-1} - y_n\| \\
&< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \frac{1}{2^{n+1}} \\
&< \frac{1}{2^m} + \frac{1}{2^{m+1}} + \ldots + \\
&= \frac{\frac{1}{2}^m}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{m+1}}
\end{aligned}$$

So $\{y_n\}$ is a Cauchy sequence in N. Since N is complete, there exists a vector y in N such that $y_n \to y$. Finally

$$\begin{aligned}
\|(x_n + M) - (y + M)\| &= \|x_n y + M\| \\
&\le \inf\{\|x_n - y + m\|; m \in M\} \\
&\le \|x_n - m + y\| \text{ for all } m \in M
\end{aligned}$$

But $y_n = x_n + m_n$ for some $m_n \in M$

$$\le \|y_n - y\| \to 0 \text{ since } y_n \to y.$$

Hence $x_n + M \to y + M \in N/M$

$\Rightarrow$ N/M is complete.

Definition 1.20. A series $\sum_{n=1}^{\infty} a_n, a_n \in X$ is said to be convergent to $x \in X$, where X is a normed linear space if the sequence of partial sums $< S_n >$ where $S_n = \sum_{i=1}^{n} a_i$ converges to x i.e. for every $\in > 0$, there exists $n_0 \in N$ such that $\|S_n - x\| < \in$ for $n \ge n_0$. A series$\sum_{n=1}^{\infty} a_n$ is said to be absolutely convergent if $\sum_{n=1}^{\infty} \|a_n\|$ is convergent.

Since every normed linear space is a metric space, hence every convergent sequence in it is Cauchy but not conversely.

The following theorem gives a nice characterization of a Banach space in terms of series.

Theorem 1.21. A normed linear space is complete if and only if every absolutely convergent series in X is convergent.

Proof: Let X be complete. For each positive integer n, let x_n be an element of X such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Let $y_k = \sum_{n=1}^{k} x_n$. Then

$$\|y_{p+k} - y_k\| = \|\sum_{n=1}^{k+p} x_n - \sum_{n=1}^{k+p} x_n\|$$

$$= \|\sum_{n=k+1}^{k} x_n\|$$
$$\leq \sum_{n=k+1}^{k} \|x_n\| \to 0 \,\text{as}\, k \to \infty.$$

Hence $< y_k >_{k=1}^{\infty}$ is a Cauchy sequence in X and since X is complete, there exists $x \in X$ such that

$$x = \lim_{k\to\infty} y_k = \lim_{k\to\infty} \sum_{n=1}^{k} x_n = \sum_{n=1}^{\infty} x_n$$

Thus the series $\sum_{n=1}^{\infty} x_n$ converges.

Conversely, let every absolutely convergent series be convergent. Let $< x_n >$ be a Cauchy sequence in X.
For each positive integer k, $\exists$ a positive integer n_k such that

$$\|x_n - x_m\| < \frac{1}{2^k} \text{ for all } m, n \geq n_k.$$

From this, we get
$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k} \text{ for all } k = 1, 2, 3, \cdots.$$
Now, the series
$$\sum_{1}^{\infty} \|x_{n_{k+1}} - x_{n_k}\| \leq \sum_{1}^{\infty} \frac{1}{2^k} < \infty.$$
By the hypothesis, $\sum_{k=1}^{\infty}(x_{n_{k+1}} - x_{n_k})$ has a convergent subsequence $< x_{n_k} >$ and so the whole sequence $< x_n >$ converges. Hence X is complete.

Riesz Lemma

Let X be a proper closed linear subspace of a normed linear space Xover the field K. Let $0 < \alpha < 1$, then $\exists\, x_\alpha \in X$ such that

$$\|x_\alpha\| = 1 \text{ and } \inf_{y\in Y} \|x_\alpha - y\| \geq \alpha.$$

Theorem 1.22. Let X be normed linear space. The closed unit ball

$$B = \{x \in X; \|x\| \leq 1\}$$

in X is compact if and only if X is finite dimensional.

Proof: Let X be finite dimensional. Since B is closed and bounded. It follows from Heine-Borel theorem that it is compact.

Conversely suppose that B is compact but X is infinite dimensional. Choose $x_1 \in X$ with $\|x_1\| = 1$. This x_1 generates a one-dimensional subspace X_1 of X.

Since every finite dimensional subspace of a normed linear space is closed, it follows that X_1 is closed. Now X_1 is a proper subspace of $X$and dim $X = \infty$.

By Riesz-Lemma there is an $x_2 \in X$ of norm 1 such that

$$\|x_2 - x_1\| \geq \frac{1}{2}.$$

The set $\{x_1, x_2\}$ generates a two dimensional proper closed subspace X_2 of X.

By Riesz Lemma, there is an x_3 of norm 1 such that for all $x \in X_2$, we have

$$\|x_3 - x\| \geq \frac{1}{2}$$

In particular

$$\|x_3 - x_1\| \geq \frac{1}{2}$$

and

$$\|x_3 - x_2\| \geq \frac{1}{2}.$$

Proceeding by induction, we obtain a sequence $< x_n >$ of elements of B such that

$$\|x_m - x_n\| \geq \frac{1}{2}(m \neq n)$$

i.e. $\{x_n\}$ can not have a convergent subsequence which contradicts the compactness of B. Hence the result.

Cauchy's inequality. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two n-tuples of real or complex numbers. Then

$$\sum_{i=1}^{n} |x_i y_i| \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}$$

Proof: We first remark that if a and b are any two non-negative real numbers, then $a^{1/2}.b^{1/2} \leq \frac{a+b}{2}$. Infact, on squaring both sides and rearranging, it is equivalent to $0 \leq (a-b)^2$ which is obviously true. If $x = 0$ or $y = 0$, the assertion is clear. We therefore assume that $x \neq 0$ or $x \neq 0$ We define a_i and b_i by

$$a_i = \left[\frac{|x_i|}{\|x\|}\right]^2 \text{and} b_i = \left[\frac{|y_i|}{\|y\|}\right]^2.$$

Since $a^{1/2}.b^{1/2} \leq \frac{a+b}{2}$.

$$\Rightarrow \frac{|x_i y_i|}{\|x\|.\|y\|} \leq \frac{|x_i|^2/\|x\|^2 + |y_i|^2/\|y\|^2}{2}$$

Summing these inequalities as i varies from 1 to n, we obtain

$$\frac{\sum_{i=1}^{n} |x_i y_i|}{\|x\|.\|y\|} \leq \frac{1+1}{2} = 1$$

and hence

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|.\|y\|$$

which proves $\sum_{i=1}^{n} |x_i y_i| \leq \|x\|.\|y\|$.

Minkowski's-inequality: Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be two n-tuples of real or complex numbers. Then

$$\left[\sum_{i=1}^{n} |x_i + y_i|^2\right]^{\frac{1}{2}} \leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}}.$$

or

$$\|x+y\| \leq \|x\| + \|y\|.$$

Proof: Using Cauchy's inequality, we have the following chain of relations.

$$\begin{aligned}
\|x+y\|^2 &= \sum_{i=1}^{n} |x_i + y_i|.|x_i + y_i| \\
&\leq \sum_{i=1}^{n} |x_i + y_i|(|x_i + y_i|) \\
&= \sum_{i=1}^{n} |x_i + y_i|.|x_i| + \sum_{i=1}^{n} |x_i + y_i|.|y_i| \\
&\leq \|x+y\|.\|x\| + \|x+y\|.\|y\| \\
&= \|x+y\|(\|x\| + \|y\|)
\end{aligned}$$

If $\|x+y\| = 0$, the inequality to be proved is trivially true. If $\|x+y\| \neq 0$, then dividing the inequality (1) through by $\|x+y\|$, we obtain

$$\|x+y\| \leq \|x\| + \|y\|.$$

and Minkowski inequality is established.

Example 1.23. Let p be a real number such that $1 \leq p < \infty$. We denote by l_p^n, the space of all $n-$ tuples $x = (x_1, x_2, \ldots, x_n)$ of scalars with the norm defined by

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}}$$

Since the norm defined in the last example is obviously the special case of this norm which corresponds to $p = 2$, so the real and complex spaces l_2^n are the n-dimensional Euclidean and unitary spaces R^n and C^n. Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ and let α be any scalar. Then l_p^n is a linear spaces with respect the operations

$$x + y = (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

and

$$\alpha x = \alpha(x_1, x_2, \cdots, x_n) = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n).$$

Since the norm introduced above is non-negative and absolute homogeneous, so to show that I_p^n is a normed linear space, it is sufficient to prove that

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

To show this, we first establish the following inequalities.

Holder's inequality. Let p and q be real numbers greater than 1, with the properties that $\frac{1}{p} + \frac{1}{q} = 1$ (Such numbers are called conjugate indices). Then for any complex number

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n).$$

$$\sum_{i=1}^{n} |x_i y_i| \leq \left[\sum_{i=1}^{n} |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} |y_i|^q\right]^{\frac{1}{q}}$$

or in our notations

$$\sum_{i=1}^{n} |x_i y_i| \leq \|x\|_p . \|y\|_q$$

Proof: If $x = 0$ or $y = 0$ the inequality is obvious. So assume that both are non-zero. Set

$$a_i = \left[\frac{x_i}{\|x\|_p}\right]^p \quad \text{and} \quad b_i = \left[\frac{y_i}{\|y\|_q}\right]^q$$

Then using

$$a_i^{1/p} b_i^{1/q} \leq \frac{a_i}{p} + \frac{b_i}{q} \qquad (a, b \geq 0)$$

We have

$$\frac{|x_i y_i|}{\|x\|_p \|y_q\|} \leq \frac{a_i}{p} + \frac{b_i}{q}$$

or or

$$\frac{|x_i y_i|}{\|x\|_p \|y_q\|} \leq \frac{1}{p}\frac{|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{|y_i|^q}{\|y\|_q^q}$$

Summing these inequalities as i varies from 1 to n, we have

$$\begin{aligned}
\frac{\sum_{i=1}^n |x_i y_i|}{\|x\|_p \|y\|_{\ q}} &\leq \frac{1}{p}\frac{\sum_{i=1}^n |x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{\sum_{i=1}^n |y_i|^q}{\|y\|_q^p} \\
&= \frac{1}{p}\frac{(\|x\|_p)^p}{\|x\|_p^p} + \frac{1}{q}\frac{(\|y\|_q)^q}{\|y\|_q^q} \\
&= \frac{1}{p} + \frac{1}{q} = 1
\end{aligned}$$

$$\Rightarrow \qquad \sum_{i=1}^n |x_i y_i| \leq \|x\|_p . \|y\|_q$$

We notice that when $p = q = 2$. Holder's inequality converts into Cauchy's inequality.

Minkowski's inequality: Let p be a real number such that $p \geq 1$. Then for any complex numbers

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad y = (y_1, y_2, \ldots, y_n)$$

$$\left[\sum_{i=1}^n |x_i + y_i|^p\right]^{\frac{1}{p}} = \left[\sum_{i=1}^n |x_i|^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^n |y_i|^p\right]^{\frac{1}{p}}$$

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p$$

Proof: The inequality is trivial when $p = 1$. So assume $p > 1$. Using Holder's inequality, we obtain

$$\begin{aligned}
|x_i + y_i|_p^p &= \sum_{i=1}^n |x_i + y_i|^p \\
&= \sum_{i=1}^n |x_i + y_i| . \|x_i + y_i\|^{p-1} \\
&\leq \sum_{i=1}^n |x_i||x_i + y_i|^{p-1} + \sum_{i=1}^n |y_i||x_i + y_i|^{p-1} \\
&\leq (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |x_i + y_i|^{(p-1)q})^{\frac{1}{q}} + (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^n |x_i + y_i|^{(p-1)q})^{\frac{1}{q}}
\end{aligned}$$

Since $(p-1)q = p$, we have

$$= (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^p)^{\frac{1}{p} \cdot \frac{p}{q}} + (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{n} |x_i + y_i|^p)^{\frac{1}{p} \cdot \frac{p}{q}}$$
$$= (\|x\|_p + \|y\|_p).(\|x+y\|_p^{p/q})$$

If $\|x+y\|_p = 0$, then the result is trivial. If $\|x+y\|_p \neq 0$, then dividing inequality (1), throughout by $\|x+y\|_p^{p/q}$ we obtain

$$\frac{\|x+y\|_p^p}{(\|x+y\|_p^{p/q})} \leq (\|x\|_p + \|y\|_p).(\frac{\|x+y\|_p^{p/q}}{\|x+y\|_p^{p/q}})$$
$$\Rightarrow \quad \|x+y\|_p^{p-\frac{p}{q}} \leq \|x\|_p + \|y\|_p$$
$$\Rightarrow \quad \|x+y\|_p^{p(1-(\frac{1}{q}))} \leq \|x\|_p + \|y\|_p$$
$$\Rightarrow \quad \|x+y\|_p^1 \leq \|x\|_p + \|y\|_p$$

since $\frac{1}{p} + \frac{1}{a} = 1 \Rightarrow \frac{1}{p} = 1\frac{1}{q}$

Thus $\|x+y\|_p \leq \|x\|_p + \|y\|_p$.

In view of the Minkowski's inequality, it follows that l_p^n is a normed linear space as triangle inequality can be established using this

CHAPTER 2

Banach Spaces and Its examples.

Particularly useful and important metric spaces are obtained if we take a vector space and define on it a metric by means of a norm. The resulting space is called normed linear space. If it is a complete a metric space, it is Banach space. The theory of normed spaces, in particular Banach space, and the theory of linear operators defined on them are the most highly developed parts of functional analysis. The present chapter is devoted to the those basic ideas of those theories.

Definition 2.1. A metric space (X,d) is said to be complete $\Leftrightarrow$ every Cauchy sequence in X has convergent subsequence. Or every Cuachy sequence of points of X converges to some point of X.

Banach Space:- A complete normed linear space is called Banach space OR A normed space $(E,\|.\|)$ over field K is called Banach space over K if E is complete metric arising from norm.

Example 2.2. Show that linear space R^n or C^n of all n-tupples $x=(x_1,x_2,\cdots,x_n)$ of real or conplex numbers are Banach space w.r.t. norm

$$\|x\| = \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} \tag{2}$$

Solution. First, we show that R^n or C^n are normed linear space w.r.t. norm 2.

(i) $\|x\| \geq 0$ for each $x_i \geq 0$

(ii) $\|x\| = 0 \Leftrightarrow x = 0$

For $\|x\| = 0 \Leftrightarrow \left[\sum_{i=1}^{n} |x|^2\right]^{\frac{1}{2}} = 0 \Leftrightarrow \sum_{i=1}^{n} |x|^2 = 0 \Leftrightarrow |x_i| = 0 \Leftrightarrow x_i = 0$ for all i $\Leftrightarrow$ $x = (x_1,x_2,\cdots,x_n) = 0$

(iii) $\|\alpha x\| = \left[\sum_{i=1}^{n} |\alpha x|^2\right]^{\frac{1}{2}} = \left[\sum_{i=1}^{n} |\alpha|^2|x|^2\right]^{\frac{1}{2}} = |\alpha|\left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} = |\alpha|\|x\|$

(iv) Sub-additivity

$$\begin{aligned}\|x+y\| &= \left[\sum_{i=1}^{n} |x_i+y_i|^2\right]^{\frac{1}{2}}\\ &\leq \left[\sum_{i=1}^{n} |x_i|^2\right]^{\frac{1}{2}} + \left[\sum_{i=1}^{n} |y_i|^2\right]^{\frac{1}{2}} \text{[Using Minkowski's Inequality]}\end{aligned}$$

Further, we shal prove that R^n or C^n are complete. For this let $< x_n >$ be Cauchy sequence in R^n or C^n.

These spaces are metric space w.r.t. metric d defined by $d(x,y) = \|x-y\|$. Since each x_n is n-tupple of real (or complex) numbers and hense $x_n = (x_{n1}, x_{n2}, \cdots, x_{nn})$, By defintion of Cauchy sequence, given $\in > 0$, $\exists$ positive interger n_0 such that for all $l, m \geq n_0$

$$\Rightarrow \|x_n - x_l\| < \in \Rightarrow \ \|x_n - x_l\|^2 < \in^2 \Rightarrow \sum_{i=1}^{n} |x_{mi} - x_{li}|^2 < \in^2$$

$$\Rightarrow |x_{mi} - x_{li}|^2 < \in^2 \text{ for all } i. \ \Rightarrow |x_{mi} - x_{li}| < \in \text{ for all } i.$$

This shows that $< x_{mi} >$ is Cauchy sequence for all i. Hense R^n or C^n is complete and therefore Banach Spaces.

Example 2.3. Show that l_p^n is Banach Space.

Solution: Here we prove completeness of l_p^n. For this let $< x_m >$ be a Cauchy sequence in l_p^n. We write

$$x_m = (x_1^m, x_2^m, \ldots, x_n^m)$$

Let $\in > 0$ be given, since $< x_m >$ is a Cauchy sequence, there exists $a + ve$ integer m_0 such that

$$\begin{aligned}
& \|x_m - x_l\|_p < \in \quad \text{whenever } m, l \geq m_0 \\
\Rightarrow \quad & \|x_m - x_l\|_p^p < \in^p \\
\Rightarrow \quad & \sum_{i=1}^{n} |x_i^{(m)} - x_i^{(l)}|^p < \in^p \qquad (1) \\
\Rightarrow \quad & |x_i^{(m)} - x_i^{(l)}|^p < \in^p, \quad i = 1, 2, \ldots, n \\
\Rightarrow \quad & |x_i^{(m)} - x_i^{(l)}| < \in
\end{aligned}$$

This shows that the sequence $< x_i^m >_{m=1}^{\infty}$ is a Cauchy sequence in C or R and completeness of R and C implies that each of these sequence converges to a point say z in Cor R such tha

$$\lim_{m\to\infty} x_i^{(m)} = z_i \quad (i = 1, 2, \ldots, n) \qquad (3)$$

We will now show that the Cauchy sequence $< x_m >$ converges to the point $z = (z_1, x_2, \ldots, z_n) \in l_p^n$. To prove this let $l \to \infty$ in 1, they by 3 for $m \geq m_0$, we have

$$\sum_{i=1}^{n} |x_i^{(m)} - z_i|^p < \in^p \Rightarrow \|x_m - z\|_p^p < \in^p$$

$\Rightarrow \quad \|x_m - z|_p <\in$

Consequently the Cauchy sequence $< x_m >$ converges to $z \in l_p^n$. Hence l_p^n is complete and therefore it is a Banach space.

Example 2.4. Let p be a real number such that 1_p denote the space of all sequences $x =< x_1, x_2, \cdots, x_n, \cdots >$ of scalars S such that $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

Show that 1_p is *a* Banach space under the norm

$$\|x\|_p = [\sum_{i=1}^{n} |x_n|^p]^{\frac{1}{p}}$$

Solution: $[N_1]$: Since each $\sum_{i=1}^{n} |x_n|^p \geq 0 \Rightarrow$ we have $\|x\|_p \geq 0$ and $\|x\|_p = 0 \Leftrightarrow [\sum_{n=1}^{\infty} |x_n|^p]^{\frac{1}{p}} = 0 \Leftrightarrow \sum_{n=1}^{\infty} |x_n|^p = 0$

$\Leftrightarrow |x_n|^p = 0 \quad \forall\, n = 1, \ldots, \infty$

$\Leftrightarrow x_n = 0 \quad \forall\, n = 1, \ldots, \infty$

$\Leftrightarrow x =< x_1, x_2, \ldots, x_n, \ldots >= 0$

$[N_2]$ is $\qquad |x + y|_p \leq \|x\|_p + \|y\|_p$

$$\Rightarrow \qquad \|x + y\|_p = [\sum_{n=1}^{\infty} |x_n + y_n|^p]^{\frac{1}{p}} \quad (1 \leq p \leq \infty)$$

$$\leq [\sum_{n=1}^{\infty} |x_n|^p]^{\frac{1}{p}} + [\sum_{n=1}^{\infty} |y_n|^p]^{\frac{1}{p}}$$

[Minkowski's inequality for sequence]

$[N_3] \qquad \|\alpha x\|_p = [\sum_{i=1}^{\infty} |\alpha x_n|^p]^{\frac{1}{p}}$

$$= [\sum_{n=1}^{\infty} |\alpha|^p |x_n|^p]^{\frac{1}{p}}$$

$$\|\alpha\| = [\sum_{n=1}^{\infty} |x_n|^p]^{\frac{1}{p}} = |\alpha| . \|x\|_p.$$

Thus l_p is a normed linear space.

To prove that l_p is complete.

Let $< x_n >_{n=1}^{\infty}$ be a Cauchy sequence in l_p. Since each x_n is itself a sequence of scalars. We shall denote an element x_m by

$$x_m =< x_1^{(m)}, x_2^{(m)}, \ldots, x_n^{(m)}, \ldots >$$

Where $\sum_{n=1}^{\infty} |x_n^{(m)}|^p < \infty$. Since each $< x_n >$ is a cauchy sequence in l_p, given $\in > 0, \exists a + ve$ integer m_0 such that $n, m \geq m_{0.}$

$$\Rightarrow \|x_n - x_m\|_p < \in \tag{4}$$

In particular

$$n \geq m_0 \Rightarrow \|x_n - m_0\|_p < \in \tag{5}$$

Thus if $n \geq m_0$, then

$$\|x_n\|_p = \|x_n - x_{m0} + x_{m0}\|_p \leq \|x_n - x_{m0}\|_p + \|x_{m0}\|_p < \in + \|x_{m0}\|_p$$

If $\in + \|x_{m_0}\|_p = A$ so that $A >, 0$
Then

$$[\sum_{n=1}^{\infty} |x_n^m|^p]^{\frac{1}{p}} < A$$

$$\|x_n\|_p < A \quad \text{for } A \geq m_0. \tag{6}$$

As in the above examples, from 4, it can be shown that for fixed i, the sequence $< x_i^n >_{n=1}^{\infty}$ is a Cauchy sequence in C or R and consequently it must converge to a number say z_i.

Let $z =< z_1, z_2, \cdots, z_n >$ we assert that $z \in 1_p$ and the cauchy sequence $< x_n >$ converges to $z \in 1_p$ and we first show that $z \in 1_p$, from 6 we have for $n \geq m_0$

$$\|x_n\|_p^p < A^p \Rightarrow \sum_{i=1}^{\infty} |x_i^{(n)}|^p < A^p$$

Hence for any +ve integer L, we have

$$\sum_{i=1}^{L} |x_i^{(n)}|^p \quad (n \geq m_0) \tag{7}$$

But for $i=1,\ldots,L$, we have $x_i^{(n)} \to z_i$ as $n\to\infty$. Hence letting $n\to\infty$. in 7, we obtain

$$\sum_{i=1}^{L} |z_i|^p \leq A^p \quad (L=1,2,\ldots)$$

$$\Rightarrow \qquad \sum_{i=1}^{\infty} |z_i|^p \leq A^p < \infty$$

This proves that $z = < z_n >_{n=1}^{\infty} \in l_p$.

Finally from 4, for $n,m \geq m_0$

$$\|x_n - x_m\|_p^p < \in^p \Rightarrow \sum_{i=1}^{\infty} |x_i^{(n)} - x_i^{(m)}|^p < \in^p$$

Hence for any $+ve$ integer L,we have

$$\sum_{i=1}^{L} |x_i^{(n)} - x_i^{(m)}|^p < \in^p \quad (n,m \geq m_0)]$$

Letting $m\to\infty$ and using $\lim_{m\to\infty} x_i^{(m)} = z_i$ we obtain

$$\Rightarrow \quad \sum_{i=1}^{L} |x_i^{(n)} - z_i|^p < \in^p \quad \text{for all } n \geq m_0$$

Example 2.5. (The space l_2). Let l_2 denote the linear space of all sequences $x = < x_1, x_2, \ldots >$ of all scalars such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty$$

Show that l_2 is a Banach space under the norm $\|x\| = [\sum_{n=1}^{\infty} |x_n|^2]^{\frac{1}{p}}$.

Solution: This space is called Hilbert coordinate space or sequence space. This is a particular case of the previous example with $p=2$. If the scalars are real, then l_2 is known as infinite dimensional Euclidean space and is denoted by R^{∞}. If the scalars are complex, then $l_2$is called infinite dimensional unitary space denoted by C^{∞}.

Example 2.6. Lex X be non-zero normed linear space. Prove that X is Banach space iff $\{x \in X : \|x\| = 1\}$ is complete. Since

Solution. Let X be Banach space and $M = \{x \in X : \|x\| = 1\}$. Since X is Banach space $\Rightarrow X$ is complete as a metric space. Let $< x_n >$ be Cauchy sequence in M, then $\|x_n\| = 1$ for all n. Since $< x_n >$ is a Cauchy sequence in $M \subset X \quad \Rightarrow < x_n >$ is a Cauchy sequence in X, but X is complete.
$\Rightarrow \ \exists\, x \in X$ such that $x_n \to x$.
Again since, $\|.\|$ is continuous function, Therefore, we have

$$\|x_n\| \ \to \ \|x\| \text{ or } \|x\| = \lim_{n\to\infty} \|x_n\| = 1 \text{ or } \|x\| = 1. \qquad \text{[By definition of M]}$$

It follows that $x \in M$. Thus Cauchy sequnce $< x_n >$ of points in M converges to a point $x \in M$. Hence M is complete.

Converse. Let M is complete, we shall prove that X is Banach space. Since X is normed linear space, so it only require to prove that X is complete. For this let $< x_n >$ be a Cauchy sequence in X, then by definition

$$\|x_n - x_m\| \ \to \ 0 \,\text{as}\, m, n \ \to \ 0. \tag{8}$$

But, By 8 $\Rightarrow \leq \|x_n - x_m\| \ \to \ 0$.
Which shows that $< \|x_n\| >$ is a Cauchy sequence in R, being complete. Hence $\exists\, \alpha \in R$ such that

$$\|x_n\| \ \to \ \alpha \,\text{as}\, n \ \to \ \infty. \tag{9}$$

Write

$$y_n = \frac{x_n}{\|x_n\|} \text{ for all } n \in N. \tag{10}$$

We shall now show that $< y_n >$ is a Cauchy sequence, for this

$$\begin{aligned}
\|y_n - y_m\| &= \left\| \frac{x_n}{\|x_n\|} - \frac{x_m}{\|x_m\|} \right\| = \left\| \frac{x_n}{\|x_n\|} + \frac{x_m}{\|x_m\|} - \frac{x_m}{\|x_m\|} - \frac{x_m}{\|x_m\|} \right\| \\
&\leq \frac{\|x_n - x_m\|}{\|x_n\|} + \frac{\|x_m\|.\|(\|x_n\| - \|x_m\|)\|}{\|x_n\|.\|x_m\|} \\
&\leq \frac{\|x_n - x_m\|}{\|x_n\|} + \frac{\|(x_n - x_m)\|}{\|x_n\|}
\end{aligned}$$

or

$$\|y_n - y_m\| \leq 2.\frac{\|(x_n - x_m)\|}{\|x_n\|}$$

Letting the limit as $n,m \to \infty$, we have

$$\|y_n - y_m\| \le 0 \text{ but } \|y_n - y_m\| \le 0$$

$\Rightarrow \quad \|y_n - y_m\| = 0$ as $n,m \to \infty$.
$\Rightarrow \quad < y_n >$ is Cauchy sequence in X. Since, from 10, $\|y_n\| = 1$ for all n and $y_n \in M$. Which Finally, implies that $< y_n >$ is Cauchy sequence in M, which is being complete. Hense $\exists\, y \in M$ such that $y_n \to y$ as $n \to \infty \Rightarrow \frac{x_n}{\|x_n\|} \to y$ as $n \to \infty$. or by 9 $\Rightarrow \quad \lim_{n\to\infty} x_n = \alpha y$. But as $y \in M$, α a scalar, and M is linear space $\Rightarrow \alpha y \in M \subset X \Rightarrow \alpha y \in X$. therefore a Cauchy sequence of points in X converges to a point in X. Thus X is complete, so a Banach Space.

Example 2.7. Let p be a positive real number. A measurable function f defined on $[0,1]$ is said to belong to the space $L^p[0,1]$ if $\int_0^1 |f|^p < \infty$.

Thus L^1 consists precisely of the L ebesgue integrable functions on $[0,1]$. Since $|f+g|^p \le 2^p(|f|^p + |g|^p)$, it follows that $f+g \in L^p$ if $f,g \in L^p$. Also αf is in L^p, therefore $\alpha f + \beta g \in L^p$ whenever $f,g \in L^p$. For a function f in L^p, we define

$$\|f\| = \|f\|_p = (\int_0^1 |f|^p)^{\frac{1}{p}}$$

we observe that $\|f\| = 0 \Leftrightarrow f = 0$ almost everywhere. Thus one of the requirement for a space to be a normed linear space is not satisfied. To overcome this difficulty, we consider two measurable functions to be equivalent if they are equal almost every where. If we do not distinguish between equivalent functions, then L^p space shall become a normed linear space. Thus we should say that the elements of L^p are not functions but rather equivalence classes of functions. If α is a constant, then $\|\alpha f\| = |\alpha|.\|f\|$. Thus to show that the linear space L^p is normed linear space, it is sufficient to show that $\|f+g\| \le \|f\| + \|g\|$. To show this again we establish two inequalities:

Holder's Inequality.

Theorem 2.8. If p and q are non-negative extended real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and if $f \in L^p$, and $g \in L^q$, then

$$\int |fg| \le \|f\|_p.\|g\|_q$$

Proof. The case $p = 1$ and $q = 1$ is straight forward. We assume therefore that $1 < p < \infty$ and consequently $1 < q < \infty$. Let us first suppose that

$\|f\|_p = \|g\|_q = 1$. Using the inequality

$$\alpha^{\lambda}\beta^{1-\lambda} \leq \lambda\alpha + (1-\lambda)\lambda, \; \alpha \text{ and } \beta \text{ are non-negative reals.}$$

Taking $\alpha = |f(t)|^p$, $\beta = |g(t)|^q$

$$\lambda = \frac{1}{p}, \;\; 1-\lambda = 1-\frac{1}{p} = \frac{1}{q},$$

we obtain

$$|f(t).g(t)| \leq \frac{1}{p}|f(t)|^p + \frac{1}{q}|g(t)|^q$$

Now integration yields

$$\int |fg| \leq \frac{1}{p}\int |f|^p + \frac{1}{p}\int |g^q| = 1$$

If $\|f\| = 0$ or $\|f\| = 0$, then the inequality to be established is trivial. Let f and g be any elements of L^p and L^p with $\|f\| \in 0$. Then $\frac{f}{\|f\|}_p$ and $\frac{g}{\|g\|}_q$ both have norm 1. Substituting them in (1) gives

$$\frac{1}{\|f\|_p\|g\|_q}\int |fg| = \int \frac{|f|}{\|f\|_p}.\frac{|g|}{\|g\|_q} \leq 1$$

and hence

$$\int |fg| \leq \|f\|_p.\|q\|_q$$

Minkowski's Inequality.

Theorem 2.9. If f and g are in L^p, then so is $f+g$ and

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

Proof. Since $|f+g|^p \leq 2^p(|f|^p + |g|^p)$, therefore $f, g \in L^p$ implies $f = g \in L^p$, the inequality is clear when $p = 1$, so we assume that $p > 1$. Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $(p-1)q = p$. Also

$$\int |f+g|^p \leq \int |f+g|.|f+g|^{p+1}$$

$$\Rightarrow \quad \int |f+g|^p \leq \int |f|.|f+g|^{p-1} + \int |g|.|f+g|^{p-1} \tag{11}$$

We note that

$$\int [|f+g|^{p-1}]^q = \int |f+g|^{(p-1)q} = \int |f+g|^p < \infty \qquad \text{(since } pq - q = p\text{)}$$

Therefore $|f+g|^{p-1} \in L^q$. Since $f, g \in L^p$ and we have just shown that $|f+g|^{p-1} \in L^q$, Holder's inequality (proved above) implies that $|f|.|f+g|^{p-1}$ and $|g|.|f+g|^{p-1}$ are in L^1 and

$$\int |f|.|f+g|^{p-1} \leq \|f\|_p.\|(|f+g|^{p-1})\|_q$$

$$\int |g||f+g|^{p-1} \leq \|g\|_p.\|(|f+g|^{p-1})\|_q$$

But, by definition of norm,

$$\begin{aligned}\|((|f+g|)^{p-1})\|_q &= \{\int (|f+g|)^{(p-1)q}\}^{1/q} \\ &= \{\int (|f+g|)^p\}^{1/q} \\ &= \{\|f+g\|_p^p\}^{1/q} \\ &= \{\|f+g\|_p\}^{p/q}\end{aligned}$$

Thus

$$\int |f|.|f+g|^{(p-1)} \leq \|f\|_p.\{\|f+g\|_p\}^{p/q} \tag{12}$$

$$\int |g|.|f+g|^{(p-1)} \leq \|g\|_p.\{\|f+g\|_p\}^{p/q} \tag{13}$$

Combining 11, 12 and 13, we have

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\{\|f+g\|_p\}^{p/q}$$

Dividing throughout by $\{\|f+g\|_p\},^{p/q}$ we obtain

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p$$

which completes the proof of Minkowski's inequality.

We have proved therefore that L^p space is a normed linear space. Now we prove that it is a complete space. We require some results.

Definition 2.10. A series $\sum f_n$ in a normed linear space is said to be summable to sum S if S is in the space and the sequence of partial sums of the series converges to S, that is,

$$\|S - \sum_{i=1}^{n} f_i\| \to 0$$

In such a case, we write $S\sum_{i=1}^{\infty} f_i$. The series $\sum f_n$ is said to be absolutely summable if $\sum_{i=1}^{\infty} \|fn\| < \infty$.

We know that absolute convergence implies convergence in case of series of real numbers. This is not true in general for series of elements in a normed linear space. But this implication holds if the space is complete.

Completeness of L^p (Riesz-Fisher Theorem).

Theorem 2.11. For $1 < p < \infty, L^p$-spaces are complete. or If $f_1, f_2, \ldots$ form a Cauchy sequence in L^p, that is $\|f_n - f_m\|_p \to 0$ as $n, m \to \infty$ there is an $f \in L^p$such that

$$\|f_n - f\|_p \to 0.$$

Proof: To show that the Cauchy sequence $< f_n >$ converges, we construct a subsequence of this sequence which converges almost every where on X as follows.

Since $< f_n >$ is a Cauchy sequence, then for $\in = \frac{1}{2}, \exists a + ve$ integer n_1 such that

$$n, m \geq n_1 \Rightarrow \|f_n - f_m\|_p < \frac{1}{2}$$

Similarly for $\in = (\frac{1}{2})^2$, we can choose $a + ve$ integer $n_2 > n_1$ such that $n, m \geq n_2$

$$\Rightarrow \|f_n - f_m\|_p < (\frac{1}{2})^2$$

In general having closed $n_1, \ldots, n_k$ let $n_{k+1} > n_k$ be such that

$$\|f_n - f_m\|_p < (\frac{1}{2})^{k+1}$$

for all $n, m \geq n_{k+1}$ we assert that the subsequence $< f_{nk} >_{k=1}^{\infty}$ converges almost everywhere to a limit function, $f \in L_p$.

From the construction of $< f_{nk} >$ it is evident that

$$\sum_{i=1}^{\infty} \|f_{nk-1} - f_{nk}\|_p < \sum_{i=1}^{\infty} (\frac{1}{2})^k = (\frac{\frac{1}{2}}{1 - \frac{1}{2}}) = 1 \tag{14}$$

$$g_k = |f_{n1}| + |f_{n2} - f_{n1}| + \ldots + |f_{nk+1} - f_{nk}|$$

For $k = 1, 2, 3 \ldots$. Then $< g_k >$ is an increasing sequence of non-negative measurable functions s. that

$$\begin{aligned} \|g_k^p\|_1 &= \|g_k\|_p^p \\ &= [\|\{|f_{n1}| + |f_{n2} + f_{n1}| + \ldots + |f_{nk+1} + f_{nk}|\}\|_p]^p \end{aligned}$$

$$\leq [\|f_{n1}\|_p + \sum_{i=1}^{k} \|f_{ni+1} - f_{ni}\|_p]^p \quad \text{(by Minkowski's inequality)}$$

$$\leq [\|f_{n1}\|_p + \sum_{i=1}^{k} \|f_{ni+1} - f_{ni}\|_p]^p$$

$$< [\|f_{n1}\|_p + 1]^p \quad \text{by (1)}$$

$$< \infty \quad \Rightarrow \quad \|g_k^p\|_1 < \infty$$

$$\text{or} \quad \int |g|^p du < \infty$$

Let $g = \lim_{k\to\infty} g_k$. Then by Monotone convergence theorem and the above estimate of g_k^p, we have

$$\int |g|^p du = \lim_{k\to\infty} \int |g_k^p| du < \infty$$

i.e.

$$\int [|f_{n_1}| + \sum_{i=1}^{\infty} |f_{ni+1} - f_{ni}|]^p du < \infty$$

Hence $g \in L_p$.

It follows the series

$$\sum_{i=1}^{\infty} |f_{ni+1}(x) - f_{ni}(x)|$$

converges a. e and consequently the series

$$f_{ni}(x) \sum_{i=1}^{\infty} (f_{ni+1}(x) - f_{ni}(x))$$

Converges a.e. The k^{th} partial sum of this series is $fn_{k+1}(+)$. and so the sequence $< fn_k(x) >_{k=1}^{\infty}$.

Converges to a some non-negative measurable function $f(x)$ for all $x \in A$ where A is measurable and $u(A) = 0$. Define $f(x) = 0$ for all $x \in A$. It is easy to see that f is measurable and complex valued on X.

We will now show that $f \in L_p$. Let $\in > 0$ be given. Choose l so large that

$$s,t \geq n_1 \Rightarrow \|f_s - f_t\|_p < \in$$

Then for $k \geq 1$ and $m > n_1$, we have

$$\|f_m - f_{nk}\|_p < \in \Rightarrow (\int |f_m - f_{nk}|^p du)^{\frac{1}{p}} < \in$$

$$\Rightarrow \int |f_m - f_{nk}|^p du < \in^p \ (1) \tag{1}$$

By Fatou's Lemma, we have

$$\int |f - f_m|^p du = \int \lim_{k \to \infty} |f_{nk} - f_m|^p du < \in^p \quad \text{by (2)}$$

Thus for each $m > n_{1,}$ the function $f - f_m$ is in L_p and so $f = (f - f_m) + f_m$ is also in L_p and $\lim_{n \to \infty} \|f - f_n\|_p = 0$. Thus f L_p is the limit of the sequence $< fn >$.

Hence L_p is complete.

Example 2.12. Consider the linear space of all n-tuples $x = (x_1, \ldots, x_n)$ of scalars and define the norm by

$$|x|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_i|\} \quad [\text{or } \ \sup |x_i|]$$

This space is denoted by l_∞^n.

Show that $(l_\infty^n, \|x\|\infty)$ is a Banach space. (Also called the space of bounded sequence)

Solution. We first prove that l^n is a normed linear space

$[N_1]$ Since each $|x_n| \geq 0 \Rightarrow \|x\|_\infty \geq 0$

and $\|x\|_\infty = 0 \Leftrightarrow \max\{|x_1|, |x_2|, \ldots, |x_i|\}$

$$\Leftrightarrow |x_1| = 0, |x_2| = 0, \ldots |x_n| = 0$$

$$\Leftrightarrow x_1, \ldots, x_n = 0$$

$$\Leftrightarrow (x_1, \ldots, x_n) = 0 \Leftrightarrow x = 0$$

$[N_2]$ Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

Then $\|x + y\|_\infty = \max\{|x_1 + y_1|, |x_2 + y_2|, \ldots, |x_n + y_n|\}$.

$$\leq \max\{|x_1| + |y_1|, |x_2| + |y_2|, |x_n| + |y_n|\}$$

$$\leq \max\{|x_1|, |x_2|, \ldots, |x_n|\} + \max\{|y_1|, |y_2|, \ldots, |y_n|\}$$

$$= \|x\|_\infty + \|y\|_\infty.$$

$[N_2]$ if α is any scalar, then

$$\|\alpha x\|_\infty = \max\{|\alpha x_1|, |\alpha x_2|, \ldots, |\alpha x_n|\}$$

$$= |\alpha| \max\{|x_1|, |x_2|, \ldots, |x_n|$$

$$= |\alpha|\|x\|_\infty.$$

Hence l_∞^n is a normed linear space. We now show that it is a complete space.

Let $< x_m >_{m-1}^{\infty}$ be any cauchy sequence in l_∞^n. Since each $x_m =< x_1^m, x_2^m, \ldots, x_n^m >$.

Let $\in > 0$ be given, $\exists$ a +ve integer m_0 that $l, m \geq m_0$

$$\Rightarrow \|x_m - x_l\|_\infty < \in$$
$$\Rightarrow \max\{|x_1^m - x_1^l|, |x_2^m - x_2^l|, \ldots, |x_n^m - x_n^l| < \in$$
$$\Rightarrow |x_1^{(m)} - x_1^{(l)}| < \in, \quad i = 1, \ldots, n.$$

This shows that for fixed $i, < x_i^{(m)} >_{m=1}^{\infty}$ is a Cauchy sequence of real (or complex) numbers. Since $\mathbb{C}$ or R is complete, it must converge to some $z_i \in \mathbb{C}$ or R. Thus the Cauchy sequence $< x_m >$ converges to $z_=(z_1, z_2, \ldots, z_n)$.

Rest of the proof is simple. Hence l_∞^n is a Banach space.

Show that l_∞ is a Banach space.

Example 2.13. Let $C(X)$ denote the linear space of all bounded continuous scalar valued functions defined on a topological space X. Show that $C(X)$ is a Banach space under the norm

$$\|f\| = \sup_{f \in C(x)} \{|f(x)|, x \in X\}$$

Solution: Vector addition and scalar multiplication are defined by

$$(f+g)x = f(x) + g(x), (\alpha f)x = \alpha f(x)$$

$C(X)$ is linear space under these operations. We now show that $C(X)$ is a normed linear space.

$[N_1]$ Since $|f(x)| \geq 0 \;\forall\, x \in X$, we have

$$\|f\| \geq 0$$

and $\|f\| = 0 \Leftrightarrow \sup\{|f(x)|, x \in X\} = 0$

$$\Leftrightarrow |f(x)| = 0 \quad \forall\, x \in X$$

$$\Leftrightarrow f(x) = 0 \quad \forall\, x \in X$$

$$\Leftrightarrow f = 0 \qquad \text{(zero function).}$$

$$[N_2] \quad \|f+g\| = \sup\{|(f+g)(x)|; x \in X\}$$
$$= \sup\{|f(x)+g(x)|; x \in X\}$$
$$\leq \sup\{|f(x)|+|g(x)|; x \in X\}$$
$$\leq \sup\{|f(x)|; x \in X\} + \sup\{|g(x)| x \in X\}$$
$$= \|f\| + \|g\|$$

$$[N_3] \quad \|\alpha f\| = \sup\{|(\alpha f)(x)|; x \in\}$$
$$= \sup\{|\alpha f(x)|; x \in\}$$
$$= \sup\{|\alpha||f(x)|; x \in\}$$
$$= |\alpha|. \sup\{|f(x)|; x \in\}$$
$$= |\alpha|\|f\|.$$

Hence $C(X)$ is a normed linear space. Finally we prove that $C(X)$ is complete as a metric space. Let $< f_n >$ be any Cauchy sequence in$C(X)$. Then for a given $\in > 0, \exists$ positive integer m_0 such that

$$m, n \geq m_0 \Rightarrow \|f_m - f_n\| < \in$$
$$\Rightarrow \sup\{|(f_m - fn)(x)|; x \in X\} < \in$$
$$\Rightarrow \sup\{|f_m(x) - fn(x)|; x \in X\} < \in$$
$$\Rightarrow |f_m(x) - fn(x)| < \in \forall x \in X.$$

But this is the Cauchy's condition for uniform convergence of the sequence of bounded continuous scalar valued functions. Hence the sequence $< f_n >$ must converge to a bounded continuous function on X. It follows that $C(X)$ is complete and hence it is a Banach space.

Consider linear spaces R and C of real numbers and complex numbers respectively. We introduce norm of a number x in R or C by defining $\|x\| = |x|$. Under this norm, both R and C are Banach spaces.

Consider the linear spaces R^n and CnC^n of all n tuples $x = (x_1, x_2, \ldots, x_n)$ of real and complex numbers. These spaces can be made into normed linear spaces by introducing the norm defined by $\|x\| = (\sum_{i=1}^{n} |X_i|^2)^{\frac{1}{2}}$

Exercise For Practice

Which of Following are Banach Spaces ?.

Question 2.14. The linear space $P[a,b]$ with the norm given by

$$\|x\|_\infty = Sup._{t\in[a,b]}|x(t)|$$

where $P[a,b]$ be set of all polynomials with real cofficients defined on $[a,b]$.

Question 2.15. The real linear space $C[-1,1]$ with the norm given by

$$\|x\|_1 = \int_{-1} 1|x(t)|\,dt$$

where integral is taken in the sense of Riemann.

Question 2.16. The space C of all convergent sequence $x =< \xi_i >$ with the norm given by

$$\|x\|_\infty = \sup_{1\le i<\infty} |\xi_i|$$

where integral is taken in the sense of Riemann.

Question 2.17. The space C of all sequences $x =< \xi_i >$ of bounded partial sums with the norm given by

$$\|x\|_\infty = \sup_{1\le n<\infty} \sum_{i=1}^{n} |\xi_i|$$

where integral is taken in the sense of Riemann.

Question 2.18. The linear space $C[a,b]$ with the norm given by

$$\|x\|_p = \left[\int_a^b |x(t)|^p dt\right]^{\frac{1}{p}}, 1 \le < \infty$$

Answers. Q. 2.0.14. No Q. 2.0.15. No. Q. 2.0.16. Yes Q. 2.0.17. Yes Q. 2.0.18. No.

CHAPTER 3

Space of Linear Tranformations

Our aim is to study a class of spaces which are endowed with both a topological and algebraic structure. This combination of topology and algebraic structures opens up the possibility of studying linear transformation of one such space into another. First we give some basic concepts and definitions.

Definition 3.1. Let N and N' be linear spaces with the same system of scalars. A mapping T from L into L' is called a linear transformation if

$$T(x+y) = T(x) + T(y)$$

$$T(\alpha x) = \alpha T(x)$$

or equivalently $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. Also $T(0) = T(0.0) = 0$ and $T(-x) = -T(x)$.

A linear transformation of one linear space into another is a homomorphism of first space into the second if is a mapping which preserves the linear operations.

Definition 3.2. Let N and $N^{'}$ be normed linear spaces with the same scalars and let T be a linear transformation of N into N'. We say that T is continuous, mean that it is continuous as a mapping of the metric space N into the metric space N'. [since every normed space is a metric space $d(x,y) = \|x-y\|$]. But by a result [Let X and Y be metric spaces and $f : X \to Y$. Then f is continuous $\Leftrightarrow x_n \to x \Rightarrow f(x_n) \to f(x)$.] This implies that $x_n \to x$ in $N \Rightarrow T(x_n) \to T(x)$ in N'.

In the next theorem, we convert the requirement of continuity into several more useful equivalent forms and show that the set of all continuous linear transformations of N into N' can itself be made into a normed linear space in a natural way.

Theorem 3.3. Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then the following conditions on T are equivalent to one another.

(i) T is continuous

(ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$.

(iii) $\exists$ a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$.

(iv) If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in N, then the image $T(S)$ is a bounded set in N'.

Proof. (i)⇒(ii) If T is continuous, then by the property of linear transformation we have $T(0) = 0$ and it is certainly continuous at the origin. For if T is continuous and $\{x_n\}$ is a sequence of points in N such that $x_n \to 0$, then by the continuity of T, we have

$$\begin{aligned} x_n \to 0 &\Rightarrow T(x_n) \to T(0) \\ &\Rightarrow T(x_n) \to 0 \qquad\qquad [\text{since } T(0) = 0] \end{aligned}$$

Conversely if T is continuous at the origin and $\{x_n\}$ is a sequence such that $x_n \to x$, then

$$\begin{aligned} &x_n \to x \Rightarrow x_n - x \to 0 \\ &\Rightarrow \quad T(x_n - x) \to T(0) = 0 \ \ [\text{since } T \text{ is continuous a the origin}] \\ &\Rightarrow \quad T(x_n) - T(x) \to 0 \end{aligned}$$

Hence T is continuous.

(ii)⇒(iii) Suppose that T is continuous at the origin. We shall show that a real number $K \geq 0$ such that $\|T(x)\| \leq K\|x\|$ for every $x \in N$.

We shall prove this result by contradiction. So suppose $\exists$ no such K. Therefore for each $+ve$ integer n, we can find a vector x_n. that

$$\|T(x_n)\| > n\|x_n\|$$

Which is equivalent to

$$\frac{\|T(x_n)\|}{n\|x_n\|} > 1 \quad \text{or} \quad \|T(\frac{x_n}{n\|x_n\|})\| > 1 \tag{1}$$

we put $y_n = \frac{x_n}{n\|x_n\|}$. Then $\|y_n\| = \frac{x_n}{n\|x_n\|} = \frac{1}{n} \to 0$ as $n \to \infty$.

If follows from it that $y_n \to 0$. But from (1) $T(\frac{x_n}{n\|x_n\|}) \nrightarrow o$. So T is not continuous at the origin, which is contradiction to our assumption.

Conversely, suppose that $\exists$ a real number $K \geq 0$ with the property that

$$\|T(x)\| \leq K\|x\|$$

for every $x \in N$. If $\{x_n\}$ is a sequence converging to zero, then

$$x_n \to 0 \Rightarrow \|x_n\| \to \|0\| = 0$$

Therefore $\|T(x_n)\| \leq K\|x_n\| \to 0$. Hence $T(x_n) \to 0$ which proves that T is continuous at the origin.

(iii)⇒(iv) Suppose first that $\exists$ a real number $K \geq 0$ with the property that $\|T(x)\| \leq K\|x\|$ for every $K\|x\|$ If $S = \{x : \|x\| \leq 1\}$ is the closed unit sphere in $N^{'}$, then for all x, we have

$$\|T(x)\| \leq K\|x\|$$
$$\Rightarrow \|T(x)\| \leq K, \forall\, x \in S.$$

Hence $T(S)$ is a bounded set in N'.
Conversely, suppose that $S = \{x : \|x\| \leq 1\}$is the closed unit sphere in N and $T(S)$ is bounded in $N^{'}$. Then

$$\|T(x)\| \leq K, \forall\, x \in S.$$

If $x = 0$, then $T(x) = T(0) = 0$ and therefore in this case we have clearly $\|T(x)\| \leq K\|x\|$. If $x \neq 0$, then $\frac{x}{\|x\|} \in S(\because\ \|\frac{x}{\|x\|}\| = 1)$ and therefore $\|T(\frac{x}{\|x\|})\| \leq K$
i.e. $\|T(x)\| \leq K\|x\|$.

Space of Bounded Linear Transformation

Definition 3.4. A linear transformation T is said to be bounded if $\exists$ a non-negative real number K such that

$$\|T(x)\| \leq K\|x\|, \forall\, x$$

K is called bound for T.

Remark. Thus according to the above theorem T is continuous iff it is bounded. From condition (4) of our theorem, we can define the norm of a continuous linear transformation as follows:

Definition 3.5. Let T be a continuous linear transformation, then

$$\|T\| = \sup\{\|x\|; \|x\| \leq 1\}$$

is called the norm of T. Obviously norm of T is the smallest M for which $\|T(x)\| \leq M\|x\|$ holds for every T i.e. $\|T\| = \inf\{M; \|T(x)\| \leq M\|x\|\}$.

Theorem 3.6. Let N and N' be normed linear spaces and let T be a linear transformation of N into N'. Then the T^{-1} exists and is continuous on its domain of definition iff $\exists$ exists a constant $m > 0$. that

$$m\|x\| \leq \|T(x)\| \quad \forall\, x \in N. \tag{1}$$

Proof. Let (1) hold. To show that T^{-1} exists and is continuous Now T^{-1} exists iff T is one-one. Let $x_{1,}x_2 \in N$. Then

$$\begin{aligned} T(x_1) = T(x_2) &\Rightarrow T(x_1) - T(x_2) = 0 \\ &\Rightarrow T(x_1 - x_2) = 0 \\ &\Rightarrow T\|x_1 - x_2\| = 0 \\ &\Rightarrow x_1 - x_2 = 0 \qquad \text{by (1)} \\ &\Rightarrow x_1 = x_2 \end{aligned}$$

Hence T is one-one and so T^{-1} exists. Therefore to each y in the domain of T^{-1}, $\exists$ a x in N such that

$$T(x) = y \Rightarrow x = T^{-1}(y) \tag{2}$$

Hence (1) is equivalent to

$$\begin{aligned} m\|T^{-1}y\| \leq \|y\| \Rightarrow \|T^{-1}(y)\| &\leq \frac{1}{m}\|y\| \\ &\Rightarrow T^{-1} \text{ is bounded} \\ &\Rightarrow T^{-1} \text{ is continuous (by the above theorem).} \end{aligned}$$

Conversely, let T^{-1} exists and be continuous on its domain $T(N)$. Let $x \in N$.
Since, there exists $y \in T(N)$ such that

$$T^{-1}(y) = x \Leftrightarrow T(x) = y \tag{3}$$

Again since T^{-1} is continuous, it is bounded so that there exists a +ve constant K, such that

$$\|T^{-1}y\| \leq K\|y\| \Rightarrow \|x\| \leq K\|T(x)\| \qquad \text{by (3)}$$

$$\Rightarrow m\|x\| \leq \|T(x)\| \text{ where } m = \frac{1}{K} > 0$$

Theorem 3.7. Let N and N' be normed linear spaces and let T be a bounded linear transformation of N into N'. Let

$$a = \sup\{\|T(x)\|; x \in N, \|x\| = 1\}$$

$$b = \sup\{\frac{\|T(x)\|}{\|x\|}; x \in N; x \neq 0\}$$

$$c = \inf\{K; K \geq 0; \|T(x)\| \leq K\|x\|, \forall\, x \in N\}.$$

Then

$$\|T\| = a = b = c$$

and

$$\|T(x)\| \leq \|T\|\|x\|, \forall\, x \in N.$$

Proof. By definition of norm

$$\|T\| = \sup\{\|T(x)\|; x \in N, \|x\| \leq 1\}$$

By definition of c, we have

$$\|T(x)\| \leq c\|x\|, \forall x \in N$$

and if $\|x\| \leq 1$, then

$$\|T(x)\| \leq c, \forall x \in N$$

and

$$\sup\{\|T(x)\|; x \in N, \|x\| \leq 1\} \leq c$$

i.e. $\|T\| \leq c$.
Also by definition of b and c, it is clear that $c \leq b$.
Again if $x \neq 0$, then

$$\frac{\|T(x)\|}{\|x\|} = \|T(\frac{x}{\|x\|})\|.$$

And $\frac{x}{\|x\|}$ has norm 1. Hence we conclude from the definitions of b and a that $b \leq a$. But it is evident that

$$\begin{aligned}
& a = \sup\{\|T(x)\|; x \in N, \|x\| = 1\} \leq \sup\{\|T(x)\|; x \in N, \|x\| \leq 1\} \\
\Rightarrow\quad & a \leq \|T\|.
\end{aligned}$$

Thus we have shown that

$$\|T\| \leq c \leq b \leq a \leq \|T\|$$

$$\Rightarrow \|T\| = a = b = c.$$

Finally, definition of b shows that

$$\frac{\|T(x)\|}{\|x\|} \leq \sup\{\frac{\|T(x)\|}{\|x\|}; x \in N, x \neq 0\} = b = \|T\|$$

$$\Rightarrow \quad \|T(x)\| \leq \|T\|\|x\|.$$

Remark. Now we shall denote the set of all continuous (or bounded) linear transformation of N into N' by $B(N,N')$ [where letter B stands for boundedness].

Theorem 3.8. If N and N' are normed linear spaces, then the set $B(N,N')$ of all continuous linear transformation of N into N' is itself a normed linear space with respect to the pointwise linear operations and the norm defined by

$$\|T\| = \sup\{\|T(x)\|; \|x\| \leq 1\}$$

Further if N' is a Banach space, then $B(N,N')$ is also a Banach space.

Proof. Let $B(N,N')$ be the set of bounded linear transformation on N into N'. Let $T_1, T_2 \in B(N,N')$. Define $T_1 + T_2$ by

$$(T_1 + T_2)(x) = T_1(x) + T_1(x)$$

and αT by

$$(\alpha T)(x) = \alpha T(x), \quad \forall\, x \in N.$$

It can seen that under these operations of addition and scalar multiplication, $B(N,N')$is a vector space since we know that the set S of all linear transformation from a linear space into another linear space is itself a linear space w.r.t. to the pointwise linear operations. Therefore in order to prove that $B(N,N')$ is a linear space, it is sufficient to show that $B(N,N')$ is a subspace of S. Let T_1, $T_2 \in B(N,N')$. Then T_1 and T_2 are bounded, so $\exists$ real numbers $K_1 \geq 0$ and $K_2 \geq 0$ such that

$$\|T_1(x)\| \leq K_1\|x\|$$

and

$$\|T_2(x)\| \leq K_2\|x\|$$

for all $x \in N$.
If α, β are any two scalars, then

$$\begin{aligned}
\|(\alpha T_1 + \beta T_2)(x)\| &= \|\alpha T_1(x) + (\beta T_2)(x)\| \\
&= |\alpha|\|T_1(x)\| + |\beta|\|T_2(x)\| \\
&\leq (|\alpha|K_1 + |\beta|K_2)\|x\|
\end{aligned}$$

Thus $\alpha T_1 + \beta T_2$ is bounded and so

$$\alpha T_1 + \beta T_2 \in B(N, N')$$

This proves that $B(N, N')$ is a linear subspace of S.

Now we prove that $B(N, N')$ is a normed linear space with respect to the norm defined by

$$\|T\| = \sup\{\|T(x)\|; \|x\| \leq 1]$$

which is clearly non-negative. We have

(i) $\|T\| = 0 \Leftrightarrow \sup\{\|T(x)\|; \|x\| \leq 1\} = 0$

$$\Leftrightarrow \sup\{\frac{\|T(x)\|}{\|x\|}; x \neq 0\} = 0$$

$$\Leftrightarrow \frac{\|T(x)\|}{\|x\|} = 0 \ \forall \, x \in N, x \neq 0$$

$$\Leftrightarrow \|T(x)\| = 0 \Leftrightarrow T = 0$$

(ii) $\|\alpha T\| = \sup\{\|(\alpha T)(x)\|; \|x\| \leq 1\}$

$$= \sup\{\|\alpha . T(x)\|; \|x\| \leq 1\}$$

$$= \sup\{|\alpha| \|T(x)\|; \|x\| \leq 1\}$$

$$= |\alpha| . \sup\{\|T(x)\|; \|x\| \leq 1\}$$

(iii) $\|T_1 + T_2\| = \sup\{\|(T_1 + T_2)(x)\|; \|x\| \leq 1\}$

$$= \sup\{\|T_1(x) + T_2(x); \|x\| \leq 1\}$$

$$\leq \sup\{\|T_1(x)\|; \|x\| \leq 1\} + \sup\{\|T_2(x)\|; \|x\| \leq 1\}$$

$$= \|T_1\| + \|T_2\|$$

Hence $B(N, N')$ is normed linear space. It remains to prove that if N' is a Banach space, then $B(N, N')$ is also a Banach space. For if; suppose N' is a Banach space. Then N' is complete. It sufficiency to show that is $B(N, N')$ complete. Let $\{T_n\}$ be an arbitrary Cauchy sequence in $B(N, N')$, then for any $x \in N$,

$$\begin{aligned} \|T_m(x) - T_n(x)\| &= \|(T_m - T_n)(x)\| \\ &\leq \|T_m - T_n\| \|x\| \qquad (\because \ \|T(x)\| \leq \|T\| \, \|x\|) \end{aligned} \tag{1}$$

This shows that $\{T_n\}$ is a cauchy sequence in N'. Since N' is complete, $\exists\ T(x)$ in N' such that $T_n(x) \to T_n,\ \ \forall x$. i.e.$T(x) = \lim_{n\to\infty} T_n(x)$.
Now defines a mapping T from N to $N^{'}$. It is obvious that T is linear. For

$$\begin{aligned} T(x+y) &= \lim_{n\to\infty} T_n(x+y) \\ &= \lim_{n\to\infty} T_n(x) + \lim_{n\to\infty} T_n(y) \\ &= T(x) + T(y) \end{aligned}$$

and

$$\begin{aligned} T(\alpha x) &= \lim_{n\to\infty} T_n(\alpha x) \\ &= \lim_{n\to\infty} \{\alpha T_n(x)\} \end{aligned}$$

Now $\{T_n\}$ being a Cauchy sequence, $\lim_{n\to\infty}\{\|T_{n-}T_m\|\} = 0$ and since

$$|(\|T_n\| - \|T_m\|)| \leq \|T_n - T_m\|$$

it follows that

$$\lim_{m,n\to\infty} |(\|T_n\| - \|T_m\|)| = 0$$

Therefore $\{T_n\}$ is convergent and hence bounded i.e. $\exists$ a real no. K such that

$$\|T_n\| \leq K, \quad n = 1,2,\ldots$$

and therefore

$$\|T_n(x)\| \leq \|T_n\|\|x\| \leq K\|x\|, \quad \forall\ n$$

Thus

$$\|T(x)\| = \lim_{m,n\to\infty} \|T_n(x)\| \leq K\|x\|$$

$\Rightarrow T$ is bounded.
Hence $T \in B(N,N')$. If we prove that $T_n \to T$. Then we have that $B(N,N')$ is complete. For let $\in > 0$, choose n_0 so that

$$\|T_m - T_n\| < \frac{\in}{2} \ \text{ if } \ m,n > n_0.$$

Then

$$\|T_m(x) - T_n(x)\| < \frac{\in}{2}\|x\| \ \text{ for } m,n > n_0,\ x \in N.$$

Letting $n \to \infty$, we get

$$\|T_m(x) - T_n(x)\| < \frac{\in}{2}\|x\| \text{ for } m,n > n_0,\ x \in N$$

since

$$T(x) = \lim_{m,n\to\infty} T_n(x).$$

This implies that for $m > n_0$ and $\|x\| \le 1$, we have

$$\begin{aligned}
\|T(x) - T_n(x)\| &= \|T(x) - T_m(x) + T_m(x) - -T_n(x)\| \\
&\le \|T(x) - T_m(x)\| + \|T_m(x) - T_n(x)\| \\
&\le \|T(x) - T_m\|||x|| + \|T_m - T_n\|||x|| \qquad [\because\ \|x\| \le 1] \\
&< \frac{\in}{2} + \frac{\in}{2} = \in
\end{aligned}$$

This shows that

$$\|T - T_n\| = \sup\{\|T(x) - T_n(x)\|; \|x\| \le 1\} < \in$$

Hence $T_n \to T$. Thus we have proved that $B(N,N')$ is a complete normed linear space.

Note. By the definition of bounded linear transformation, it is clear that a continuous linear transformation is bounded linear transformation and conversely. Also if N and N' are normed linear spaces, the space $L(N,N')$ or $B(N,N')$ is also called space of all continuous linear transformation. In notation if $N = N'$, the space is also denoted as $B(N)$.

Definition 3.9. A continuous linear transformation of a normed linear space into itself is called operator on N. The normed linear space consisting of all linear operators on N is denoted by $B(N)$. The above theorem asserts that if N is a Banach space then $B(N)$ **is also a Banach Space.**

Definition 3.10. An algebra is a linear space whose vectors can be multiplied in such a way that

(i) $x(yz) = (xy)z$

(ii) $x(y+z) = xy + yz$ and $(x+y)z = xz + yz$

(iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$ for all scalars α.

Thus an algebra is a linear space that is also a ring in which (iii) holds.

If the linear operators T_1 and T_2 are multiplied in accordance with the formula

$$(T_1T_2)(x) = T_1(T_2(x)), \quad \forall\, x \in N$$

Then $\beta(N)$ is a algebra in which multiplication is related to the norm by

$$\|TT'\| \leq \|T\|\|T'\|$$

This relation is proved by the following computation

$$\begin{aligned}
\|TT'\| &= \sup\{\|(TT')(x)\|; \|x\| \leq 1\} \\
&= \sup\{\|T(T'(x))\|; \|x\| \leq 1 \\
&\leq \sup\{\|T\|\|T'(x)\|; \|x\| \leq 1\} \\
&= \|T\|\{\sup \|T'(x)\|; \|x\| \leq 1\} \\
&= \|T\|\|T'\| \qquad (1)
\end{aligned}$$

Since we know that addition and scalar multiplication are jointly continuous in normed linear space, they are also jointly continuous in $\beta(N)$. Also multiplication is continuous, since if

$$T_n \to T \text{ in } B(N) \text{ and } T_n^{'} \to T^{'} \text{ in } B(N).$$

Then

$$T_nT_n' \to TT'.$$

Since

$$||T_nT_n' - TT'|| \leq \|T_n\|\|T_n' - T'\| + \|T_n - T\|\|T'\|.$$

But $\{T_n\}$ being convergent sequence in $B(N)$, it must be bounded so M such that

$$\|T_nT_n - TT'\| \leq M\|T_n' - T'\| + \|T'\|.\|T_n - T\| \to 0 \text{ as } n \to \infty.$$

We also remark that when $N \neq \{0\}$ then the identity transformation I is an identity for the algebra $\beta(N)$. In this case we clearly have

$$\|I\| = 1$$

for

$$\|I\| = \sup\{\|I(x)\|; \|x\| = 1\} = \sup\{\|x\|; \|x\| = 1\} = 1.$$

Definition 3.11. Let N and N' be normed linear spaces. A one to one linear transformation T of N into N' such that $\|T(x)\| = \|x\|$ for every x in N called isometric isomorphism. N is said to be isometrically isomorphic to N' if an isometric isomorphism of N onto N'.

Theorem 3.12. If M is a closed linear subspace of a normed linear space N and if $T : N \to N/M$ defined by $T(x) = x+M$. Show that T is continuous linear transformation for which $\|T\| \leq 1$.

Proof. Since M is closed, N/M is a normed linear space [since every closed subspace of normed space is normed] with the norm of a coset $x+M$ in N/M defined by

$$\begin{aligned}
\|x+M\| &= \inf\{\|x+m\|; m \in M\} \\
T(x_1+x_2) &= x_1+x_2+M \\
&= x_1+M+x_2+M \quad \text{[definition of } N/M\text{]} \\
&= T(x_1)+T(x_2) \\
T(\lambda x) &= \lambda x+M = \lambda(x+M) = \lambda T. \\
&\Rightarrow T \text{ is linear.} \\
\|Tx\| &= \|x+M\| = \inf\{\|x+m\|; m \in M\} \\
&\leq \inf\{\|x\|+\|m\|; m \in M\} \\
&\leq \inf\|x\|+\inf\|m\|; m \in M \\
&= \|x\|+0.
\end{aligned}$$

[since M is subspace of $N, 0$ is the element of M which has smallest norm namely zero]
Then

$$\|Tx\| \leq \|x\| \ \ \forall\, n \in N$$

$\Rightarrow T$ is bounded
Since

$$\begin{aligned}
\sup_{x\neq 0} \frac{\|Tx\|}{\|x\|} \leq 1 &\Rightarrow \|Tx\| \leq \|x\| \leq 1 \\
&\Rightarrow \sup_{x\neq 0}\{\|T(x)\|; \|x\| \leq 1\} \leq \|x\| \leq 1 \\
&\Rightarrow \|x\| \leq 1
\end{aligned}$$

Theorem 6. Let E and F be two normed linear spaces. Then they are topologically isomorphic iff $\exists\, m, M$ and a linear mapping $T : E \to F$, which is one-one and onto such that

$$m\|x\| \le \|Tx\| \le M\|x\| \quad \forall\, x \in E$$

Proof. Let E and F be topological isomorphic, then by definition $\exists$ linear mapping $T : E \to F$ such that T is continuous, bijective and T^{-1} exists and is also continuous. Then by using theorem on continuous of linear transformation, then $\exists\, M$ such that

$$\|Tx\| \le M\|x\| \quad \forall\, x \in E$$

Also by the last result, $\exists\, m > 0$ such that

$$m\|x\| \le \|Tx\| \le M\|x\|\,.$$

Since T^{-1} exists and is continuous Then we have linear one-one onto mapping such that $\exists\, m_0, M > 0$ such that

$$m\|x\| \le \|Tx\| \le M\|x\|, \;\; \forall\, x \in E$$

conversely if $\exists\, T : E \to F$ such that T is one-one onto and $\exists m, M$ such that

$$m\|x\| \le \|T(x)\| \le M\|x\|, \quad \forall\, x \in E.$$

Since

$$m\|T(x)\| \le M\|x\|.$$

Hence T is bounded.
By the theorem on continuity (or bounded) $\Rightarrow T$ is continuous.
Now from $m\|x\| \le \|T(x)\|$, T is $1-1$ and onto exists
$\Rightarrow T^{-1}$ is continuous. Hence T is bijective, continuous and T^{-1} exists and is continuous [T is open]
$\Rightarrow E$ and F are topologically isomorphic.

Remark. On a finite dimensional space C^n, or all the norms are equivalent in the sense that they define same topology up to topologycally isomorphism.

CHAPTER 4

Conjugate Spaces and Hahn Banach Theorem

This chapter deals primarily (through not exclusively) with the important class of topological vector spaces, namely, the conjugate spaces. The highlights, from the theoretical as well as the applied standpoints are the Hahn-Banach theorems with applications and Riesz-Representation theorem for bounded linear functionals on L^p

Definition 4.1. Let E and F be normed linear spaces. Then E and F are said to be equivalent normed spaces iff $\exists\ T > 0, M, m > 0$ such that

$$m\|x\| \leq \|Tx\| \leq M\|x\|, \quad \forall\, x \in E.$$

Conjugate of an Operator: Let N be a normed linear space and T a continuous linear operator on N^*, Then for any functional the composite mapping (foT) is a continuous linear functional since

$$\begin{aligned}(foT)(\alpha x+\beta y) &= f(T(\alpha x+\beta y); \quad x, y \in N\\ &= f(\alpha.T(x)+\beta T(y))\\ &= \alpha f(T(x))+\beta f(T(y))\\ &= \alpha(foT)(x)+\beta(foT)(y)\end{aligned}$$

Moreover since f and T both are continuous, foT is also continuous Hence $f \in N_*$. Define a mapping

$$T^* : N^* \to N^*$$

by

$$T^*(f) \to foT, \quad \forall\, f \in N^*.$$

This mapping is called the conjugate of the operator T.
Also we note that

$$(T^*(f))(x) = f(T(x)), \quad \forall\, x \in N.$$

We assert that T^* is linear, for

$$(T^*)(\alpha f+\beta g)(x) = (\alpha f+\beta g)(Tx)$$

$$\begin{aligned}
&= \alpha f(T(x)) + \beta .g(T(y)) \\
&= \alpha(fT)(x) + \beta(gT)(x) \\
&= \alpha(T^*(f))(x) + \beta(T^*(g))(x) \\
&= (\alpha T^*(f) + \beta T^*(g))(x)
\end{aligned}$$

T^* is also bounded (continuous) and hence

$$\begin{aligned}
\|T^*\| &= \sup\{\|(T^*f)\|; \|f\| \leq 1\} \\
&= \sup\{|T^*(f)(x)|; \|f\| \leq 1 \,\text{and}\, \|x\| \leq 1\} \\
&= \sup\{|f(T(x))|; \|f\| \leq 1 \ \|x\| \leq 1\} \\
&\leq \sup\{\|f\|\|T\|\|x\|; \|f\| \leq 1, \|x\| \leq 1\} \\
&\leq \|T\| \qquad (1)
\end{aligned}$$

Since N is a normed linear space, for a non-zero vector x in N, there exists a functional f on N such that

$$\|f\| = 1 \ \text{ and } \ f(T(x)) = \|T(x)\| \quad [\because \ \|f\| = 1 \,\text{and}\, f(x) = \|x\|]$$

$$\begin{aligned}
\|T\| &= \sup\{\|Tx\|; \|x\| \leq 1\} \\
&\leq \sup\{f(T(x)); \|x\| \leq 1 \,\text{and}\, \|f\| \leq 1\} \\
&= \sup\{|T^*(f)(x)|; \|f\| \leq 1 \,\text{and}\, \|x\| \leq 1\} \\
&= \sup\{\|(T^*f)\|\|x\|; \|f\| \leq 1 \,\text{and}\, \|x\| \leq 1\} \\
&\leq \sup\{\|(T^*f)\|; \|f\| \leq 1\} \\
&= \|T^*\| \qquad (2)
\end{aligned}$$

From (1) and (2), it follows that

$$\|T\| = \|T^*\| \qquad (3)$$

consider the mapping

$$\phi : B(N) \to B(N^*)$$

defined by

$$\phi(T) = T^*, \quad \forall \ T \in B(N)$$

Let $T_1, T_2 \in B(N)$. Then

$$\phi(\alpha T_1 + \beta T_2) = (\alpha T_1 + \beta T_2)^*$$

But for all $f \in N^*$ and $x \in N$, we have

$$\begin{aligned}[(\alpha T_1+\beta T_2)^*(f)](x) &= f[(\alpha T_1+\beta T_2)^*(x)]\\ &= f[\alpha T_1(x)+\beta T_2(x)]\\ &= \alpha f(T_1(x))+\beta f(T_2(x))\\ &= \alpha(fT_1)(x)+\beta(fT_2)(x)\\ &= \alpha(T_1^*(f))(x)+\beta(T_2^*(f))(x)\\ &= (\alpha[T_1^*(f)]+\beta[T_2^*(f)])(x)\\ &= \{(\alpha T_1^*+\beta T_2^*)(f)\}(x)\end{aligned}$$

Therefore, we have

$$\begin{aligned}\phi(\alpha T_1+\beta T_2) &= (\alpha T_1+\beta T_2)^*\\ &= \alpha T_1^*+\beta T_2^*\\ &= \alpha\phi(T_1)+\beta\phi(T_2),\end{aligned}$$

which shows that in ϕ linear. Also ϕ is one to one, since

$$\begin{aligned}&\phi(T_1)=\phi(T_2)\\ &\qquad \Rightarrow T_1^*=T_2^*\\ &\qquad \Rightarrow T_1^*(f)=T_2^*(f) \quad \forall\, f\in N^*\\ &\qquad \Rightarrow [T_1^*(f)](x)=[T_2^*(f)](x)\\ &\qquad \Rightarrow f(T_1(x))=f(T_2(x))\\ &\qquad \Rightarrow T_1-T_2=0\\ &\qquad \Rightarrow T_1=T_2\end{aligned}$$

Moreover

$$\|\phi(T)\|=\|T^*\|=\|T\|$$

Hence ϕ is an isometric isomorphism and it also preserves norm. If $f \in N$ and $x \in N$, then

$$\begin{aligned}((T_1T_2)^*(f))(x) &= f(T_1T_2)(x)\\ &= f(T_1(T_2(x)))\\ &= (fT_1)(T_2(x))\\ &= (T_1^*(f))(T_2(x))\end{aligned}$$

$$
\begin{aligned}
&= T_2^*(T_1^*(f))(x) \\
&= [(T_2^* T_1^*)(f)](x)
\end{aligned}
$$

i.e.

$$(T_1 T_2)^* = T_2^* T_1^*$$

and if I is an identity operator, then

$$
\begin{aligned}
[I^*(f)](x) &= f[I^*(x)] = f(x) \\
&= (I(f))(x) \\
\Rightarrow \quad I^* &= I
\end{aligned}
$$

Thus we have proved the following:

Theorem 4.2. If T is an operator on a normed linear space N, Then its conjugate T^* is defined by equation

$$[T^*(f)](x) = f[T(x)]$$

is an operator on N and the mapping T^* is an isometric isomorphism of $\mathbb{B}(N)$ into $\mathbb{B}(N^*)$ which reverses the product and preserves the identity transformation.

Theorem 4.3. A non empty subset X of a normed linear space N is bounded then $f(x)$ is a bounded set of numbers for each f in N^*.

Proof. Since $|f(x)| \leq \|f\|\|x\|$ it follows that if X is bounded, then $f(x)$ is also bounded for each f. To prove the converse, we write $X = \{x_i\}$. We now use natural imbedding $[x \to F_n]$ to map X to the subset (F_{x_i}) of N^{**}. The assumption that $f(x) = \{f(x_i)\}$ is bounded for each f implies that $\{F_{x_i}(f)\}$ is bounded for each f. Moreover since N^* is complete. The uniform boundedness theorem shows that $\{F_{x_i}\}$ is a bounded subset of N^{**}. Since natural imbedding preserves norms, therefore X is evidently a bounded subset of N^{**}.

Conjugate Spaces

We know that the spaces R and C are real and complex complete normed linear spaces. If N is an arbitrary normed linear space, then the set $B(N,R)$ or $B(N,C)$of all continuous linear transformations of N in R or C is a normed linear space. This space is called the conjugate space of N and is denoted by N^*.

The elements of N^* are called continuous linear functionals or simply functionals. The norm of a function $f \in N^*$is defined as

$$\|f\| = \sup\{|\|f(x)\|; \|x\| \leq 1|\}$$

Since R and C are Banach spaces, it follows that $B(N,R)$ and $B(N,C)$ are also Banach spaces. Thus N^* is also a Banach space.

Hahn-Banach Theorem and its applications

Hahn-Banach Theorem is a strong tool for functional analysis. In fact the theory of conjugate spaces rest on the Hahn-Banach Theorem which asserts that any linear functional on a linear subspace of a normed linear space can be extended linearly and continuously to the whole space without increasing its norm.

Theorem 4.4. Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N such that

$$f_m(x) = f(x), \quad \forall x \in M \; and \; \|f_0\| = \|f\|$$

Proof. Let f be a functional defined on a subspace M of a real normed linear space N and let x_0 be any vector of N which is not in M. Consider the set $\{M + tx_0\}$ of elements $x + tx_0$ where $x \in M$ and t is an arbitrary real number. Then $\{M + tx_0\}$ is obviously a linear manifold of N Every element of $\{M + tx_0\}$ is uniquely representable in the form $x + tx_0$, for if 0 there exists two representations $y_1 = x_1 + t_1x_0$ and $y_2 = x_2 + t_2x_0$, we can suppose that $t_1 \neq t_2$ for 0 otherwise $x_1 + t_1x_0 = x_2 + t_2x_0$ would imply $x_1 = x_2$ and the representation will be unique. Then

$$\begin{aligned} & x_1 - x_2 = (t_2 - t_1)x_0 \\ \Rightarrow \qquad & x_0 = \frac{x_1 - x_2}{t_2 - t_1} \end{aligned}$$

But this is impossible since $x_0 \in M$. and x_1, x_2. Hence $t_1 = t_2$ and Thus $x_1 = x_2$ which proves the uniqueness.

For any two elements $x_1, x_2 \in M$, we have

$$\begin{aligned} f(x_1) - f(x_2) &= f(x_1 - x_2) \\ &\leq |f(x_1 - x_2)| \\ &= \|f\|\{\|(x_1 + x_0) - (x_2 + x_0)\|\} \end{aligned}$$

$$= \|f\|\{\|x_1 + x_0\| + \|(x_2 + x_0)\|\}$$

so that

$$f(x_1) - \|f\|\|x_1 + x_0\| \leq f(x_2) + \|f\|\|x_2 + x_0\|$$

Since x_1 and x_2 are arbitrary in M,
We have

$$\sup_{x \in M}\{f(x) - \|f\|\|x + x_0\|\} \leq \inf_{x \in M}\{f(x) - \|f\|\|x + x_0\|\}$$

Thus there exists a real no which satisfies the inequality

$$\sup_{x \in M}\{f(x) - \|f\|\|x + x_0\|\} \leq \alpha \leq \inf_{x \in M}\{f(x) - \|f\|\|x + x_0\|\} \tag{1}$$

Now let y be an arbitrary element of $\{M + t\,x_0\}$. Then y is uniquely expressible in the form $y = x + tx_0$. We define a function ϕ on $\{M + t\,x_0\}$ by

$$\phi(y) = f(x) - t\alpha \;\; \forall\, y \in \{M + t\,x_0\}$$

where α is fixed real number satisfying (1). Obviously ϕ coincides with f in M and the linearity of f implies that ϕ is linear. We shall show that ϕ in bounded and has the same norm as $f(x)$. We distinguish two cases:
(i) $t > 0$. Since $\frac{x}{t} \in M$, the relation (1) yields

$$\begin{aligned}
\phi(y) &= f(x) - t\alpha \\
&= t\{f(\frac{x}{t}) - \alpha\} \\
&\leq t\{\|f\|\|\frac{x}{t}\| + x_0\} \\
&= \|f\|\|x + tx_0\| \\
&= \|f\|\|y\|
\end{aligned} \tag{2}$$

(ii) $t < 0$, In this case (i) yields

$$\begin{aligned}
f(\frac{x}{t}) - \alpha &\geq -\|f\|\|\frac{x}{t} + x_0\| \\
&= -\frac{1}{|t|}\|f\|\|y\| \\
&= \frac{1}{t}\|f\|\|y\|
\end{aligned}$$

and therefore

$$\phi(y) = f(x) - t\alpha$$

$$= t\{f(\frac{x}{t}) - \alpha\}$$
$$\leq t.\frac{1}{t}\|f\|\|y\|$$
$$= \|f\|\|y\| \tag{3}$$

Thus from (2) and (3), it follows that

$$\phi(y) \leq \|f\|\|y\| \quad \forall\, y \in \{M + tx_0\}$$

Replacing y by $-y$ in (1), we have

$$-\phi(y) \leq \|f\|\|y\| \quad \forall\, y \in \{M + tx_0\}$$

Therefore $\phi(y) \leq \|f\|\|y\|\ \forall\, y \in \{M + tx_0\}$ and therefore

$$\|\phi\| \leq \|f\| \tag{4}$$

But ϕ being an extension of f from M to $\{M + tx_0\}$ we have

$$\|\phi\| \geq \|f\| \tag{5}$$

Hence from (4) and (5)

$$\|\phi\| = \|f\|$$

Now if the elements of the set $N - M$ are arranged in transfinite sequence $x_0, x_1, x_2, \ldots, x_k, \ldots$, we extend the functional successively to the spaces

$$\{M + tx_0\} = M_0, \quad \{M_0 + tx_1\} = M_1$$

and so on since the norm remains the same at each step, continuing the above process, we arrive at a functional f_0 which satisfies both the conditions, namely

$$f_0(x) = f(x) \quad \forall\, x \in M \text{ and } \|y_0\| = \|f\|$$

This completes the proof of the theorem.

Complex Form of Hahn Banach Theorem

When N is complex and f is a complex valued function defined on M, let f_1 and f_2 be the real and imaginary parts of f. Thus for each $x \in M$, we have

$$f(x) = f_1(x) + if_2(x)$$

and

$$|f_1(x)|, |f_2(x)| \leq |f(x)| \leq \|f\|\|x\|.$$

we claim that f_1 and f_2 are real valued linear functionals.

Let $\alpha \in R$ and consider

$$|\alpha f_1(x)|, |\alpha f_2(x)| \leq |f(x)| \leq \|f\|\|x\| \tag{1}$$

Since f is a linear functional, (1) must equal

$$f_1(\alpha x) = \alpha f_1(x) \quad \text{and} \quad f_2(\alpha x) = \alpha f_2(x)$$

In a similar fashion, we can show that sums are also preserved.

Now consider

$$i(f_1(x) + i f_2(x)) = i f(x) = f(ix) = f_1(ix) + i f_2(ix)$$

Equating real and imaginary parts, we have

$$f_1(ix) = -f_2(x) \quad \text{and} \quad f_2(ix) = -f_1(x)$$

Thus

$$f(x) = f_2(x) - i f_1(x) \tag{2}$$

Now by the above proved theorem, there exists a function F_1 defined on the whole space extending f_1 such that

$$\|F_1\| = \|f_1\| \quad \text{and} \quad F_1(x) = f_1(x) \quad \forall\, x \in M.$$

We now define

$$F(x) = F_1(x) - iF_1(ix) \tag{3}$$

We now assert that F extends f. To prove this let $x \in M$ and consider (3). Since F_1 extends f_1, so

$$F_1(x) - f_1(x) \text{ and } F_1(ix) = -f_2(x)$$

Thus

$$F(x) - f_1(x) + i f_2(x) = f(x)$$

and hence F extends f.

Moreover by (3)

$$F(ix) = F_1(ix) - iF_1(i^2 x)$$

$$= F_1(ix) - iF_1(-x)$$
$$= F_1(ix) - iF_1(x)$$
$$F(ix) = [F_1(x) - iF_1(ix)]$$
$$= iF_1(x) - F_1(ix)$$

we see that
$$F(ix) = iF(x)$$
and therefore is a complex linear functional.

Put $F(x) = re^{i\theta}$, then

$$|F(x)| = |re^{i\theta}|$$
$$= r[e^{i\theta}]$$
$$= r$$

Thus $F(e^{i\theta}x)$ is a purely real quantity which implies that imaginary part of $F(e^{i\theta}x)$ i.e $F_1(e^{i\theta}x)$ must be zero.Thus
$$F(e^{i\theta}x) = F_1(e^{i\theta}x)$$

and we have

$$|F(x)| = |F_1(e^{i\theta}x)|$$
$$\leq \|F_1\|.\|x\|.|e^{i\theta}|$$
$$= \|f_1\|.\|x\|$$
$$= \|f\|.\|x\|$$

which gives $\|F\| \geq \|x\|$.

Moreover F being an extension of f, we have

$$\|F\| \geq \|f\|.$$

Hence $\|F\| = \|f\|$ and the proof is complete.

Applications of Hahn-Banach Theorem

Theorem 2. If N is a normed linear space and x_0 is a non-zero vector in N, then there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. In particular if $x \neq y(x, y \in N)$, there exists a vector $f \in N^*$ such that $f(x) \neq f(y)$.

Proof. Consider the subspace
$$M = \{\alpha x_0\}$$

consisting of all scalar multiplies of x_0 and consider the functional f defined on M as follows :

$$f : M \to F \text{ and } f(\alpha x_0) = \alpha \|x_0\|$$

Clearly, f is a linear functional with the property

$$\begin{aligned} f(x_0) &= \|x_0\| \\ \|x_0\| |f(\alpha x_0)| &= |\alpha|.\|x_0\| \\ &= \|\alpha x_0\| \qquad (1) \\ \|f\| &= \sup\{|f(\alpha x_0)|; \|\alpha x_0 \le 1\|\} \\ &= \sup\{\|\alpha x_0\|; \|\alpha x\| \le 1\} \\ &\le 1 \end{aligned}$$

But if there were a real constant k such that $k < 1$ and $|f(\alpha x_0)| \le k\|\alpha x_0\| \; \forall \; \alpha x_0 \in M$. This will contradict the equality defined by (1). Thus $\|f\| = 1$. We have thus established that f is a bounded linear functional defined on the subspace M with norm 1. Now by Hahn-Banach Theorem, the functional f can extended to a functional f_0 in N^* such that

$$f_0(x_0) = f(x_0) \quad \text{and} \quad \|f_0\| = \|f\| = 1$$

This completes the proof.
In the particular case since $x \neq y$, $x \neq 0$ and so by the above result, there exists an $f \in N^*$ such that

$$\begin{aligned} & f(x-y) = \|x-y\| \neq 0 \\ \Rightarrow \quad & f(x) - f(y) \neq 0 \\ \Rightarrow \quad & f(x) \neq f(y). \end{aligned}$$

Remark.
(1) This result shows that N^* separates the vectors of N.
(2) This result also shows that Hahn-Banach Theorem guarantee that any normed linear space has rich supply of functionals.

Theorem 4.5. Let M be a closed linear subspace of a normed linear space N and let ϕ be the natural mapping (homomorphism) of N onto N/M defined by $\phi(x) = x + M$. Show that ϕ is a continuous (or bounded) linear transformation for which

$$\|\phi\| \le 1.$$

Proof. Since M is closed and N/M is a normed linear space with the norm of a coset $x+M$ in N/M defined by

$$\|x+M\| = \inf\{\|x+m\|; m \in M\}$$

ϕ is linear: Let x, y be any two elements of N and α, β be any scalars. Then

$$\begin{aligned}
\phi(\alpha x + \beta y) &= (\alpha x + \beta y) + M \\
&= (\alpha x + M) + (\beta y + M) \\
&= \alpha(x+M) + \beta(y+M) \\
&= \alpha\phi(x) + \beta\phi(y)
\end{aligned}$$

$\Rightarrow \phi$ is linear.

ϕ is continuous:

$$\begin{aligned}
\|\phi(x)\| &= \|x+M\| \\
&= \inf\{\|x+m\|; m \in M\} \\
&\leq \|x+m\| \quad \forall\, m \in M
\end{aligned}$$

In particular for $m = 0$, we have

$$\|\phi(x)\| \leq \|x\| = 1\|x\| \quad \forall\, x \in N$$

It follows that ϕ is bounded by the bound 1 and consequently ϕ is continuous.
Further

$$\begin{aligned}
\|\phi\| &= \sup\{\|\phi x\|; x \in N; \|x\| \leq 1\} \\
&\leq \sup\{\|x\|; x \in N; \|x\| \leq 1\} \\
&\leq 1
\end{aligned}$$

Thus $\|\phi\| \leq 1$.

Theorem 4.6. Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M, then there exists a functional F in N^* such that

$$F(M) = \{0\} \quad \text{and} \quad F(x_0) \neq 0$$

Proof. Consider the natural map $\phi : N \to N/M$ defined by $\phi(x) = xM$. As shown in the last theorem it is a continuous linear transformation and if $m \in M$, then $\phi(m) = m + M = 0$, where 0 denotes the zero vector of M in N/M. In other words

$$\phi(M) = \{0\}$$

Also, since $x_0 \notin M$, we have

$$\phi(x_0) = x_0 + M \neq 0.$$

Hence by theorem 1, there exists a functional $f \in (N/M)^*$ such that

$$f(x_0 + M) = \|x_0 + M\| \neq 0$$

We now define f by $F(x) = f(\phi(x))$. Then F is a linear functional on N. With the desired properties as shown below:

F is linear:

$$\begin{aligned}
F(\alpha x + \beta y) &= f(\phi(\alpha x + \beta y)) = f(\alpha x + \beta y + M) \\
&= f(\alpha(x + M) + \beta(y + M)) \\
&= \alpha f(x + M) + \beta f(y + M) \\
&= \alpha f(\phi(x)) + \beta f(\phi(y)) \\
&= \alpha F(x) + \beta F(y)
\end{aligned}$$

F is bounded:

$$\begin{aligned}
|F(x)| &= |f(\phi(x)| \\
&\leq \|f\| \|\phi(x)\| \\
&\leq \|f\| \|\phi\| \|x\| \\
&\leq \|f\| \|x\| && [\text{since} \|\phi\| \leq 1]
\end{aligned}$$

Since f is bounded (being a functional). It follows from the above inequality that F is bounded. Thus F is a functional on N i.e. $F \in N^*$. Further if $m \in M$, then

$$F(m) = f(\phi(m)) = f(0) = 0$$

Thus

$$F(M) = 0 \; \forall m \; \in M$$

and

$$F(x_0) = f(\phi(x_0)) = f(x_0 + M) \neq 0.$$

Theorem 4.7. Let M be a closed linear subspace of a normed linear space N and let x_0 be a vector not in M. If d is the distance from x_0 to M, show that there exists a functional $f_o \in N^*$ such that

$$f_0(M) = \{0\},\ f_0(x_0) = d \quad \text{and} \quad \|f_0\| = 1.$$

Proof. Since by definition

$$d = \inf\{\|x_0 + m\|; m \in M\}$$

Since M is closed and $x_0 \notin M \Rightarrow d > 0$.
Now consider the subspace

$$M_0 = \{x + \alpha x_0; x \in M \text{ and } \alpha \text{ real}\}$$

Spanned by M and x_0. Since $x_0 \notin M$, the representation of each vector y in M_0 in the form $y = x + \alpha x_0$ is unique. For if there exists two scalars α_1 and α_2 and vectors x_1 and x_2 in M such that

$$\begin{aligned}
& y = \alpha_1 x_0 + x_1 \text{ and } y = \alpha_2 x_0 + x_2 \\
\Rightarrow \quad & (\alpha_1 - \alpha_2)x_0 = x_2 - x_1 \\
\Rightarrow \quad & x_0 = \frac{x_2 - x_1}{\alpha_1 - \alpha_2} \\
\Rightarrow \quad & x_0 \in M, \text{ which is a contradiction,}
\end{aligned}$$

since $x_0 \notin M$ by our assumption. So each y in M_0 is unique.
Define the map $f : M_0 \to R$ by

$$f(y) = \alpha d$$

where $y = x + \alpha x_0$ and d as in hypothesis. Because of the uniqueness of y, the mapping f is well defined. Also f is linear on M_0, and

$$f(x_0) = f(0 + 1.x_0) = 1.d \text{ and if } m \in M$$

then

$$f(m) = f(m + 0.x_0) = 0.d = 0$$

so that

$$f(M) = \{0\}.$$

We now prove that $\|f\| = 1$.
Since

$$
\begin{aligned}
\|f\| &= \sup_{\|y\|=1} \{\frac{|f(y)|}{\|y\|}; y \in M_0, y \neq 0\} \\
&= \sup\{\frac{|f(x+\alpha x_0)|}{\|x+\alpha x_0\|}; x \in M, \alpha \in R\} \\
&= \sup\{\frac{|\alpha d|}{\|x+\alpha x_0\|}; x \in M; \alpha \in R, \alpha \neq 0\} \\
&= \sup\{\frac{|d|}{\|x_0+\frac{x}{\alpha}\|}; x \in M, \alpha \in R, \alpha \neq 0\} \\
&= d[\inf\{\|x_0 - z\|; z \in M\}]^{-1} \\
&= d.\frac{1}{d} \\
&= 1.
\end{aligned}
$$

Thus f is a linear functional on M_0 such that

$$
f(M) = \{0\},\ f(x_0) = d \ \text{ and } \ \|f\| = 1. \tag{*}
$$

Hence by Hahn Banach Theorem, there exists a functional f_0 on the whole space N such that

$$
f(y) = f_0(y) \ \ \forall\, y \in M_0 \, \text{and} \, \|f\| = \|f_0\|
$$

Thus from (*)

$$
f_0(M) = \{0\},\ f_0(x_0) = d \, \text{and} \, \|f_0\| = 1.
$$

Riesz-Representation Theorem for Bounded Linear Functionals on L^p

Definition 4.8. A Linear functional on a real vector space $T : V \to R$, which satisfies the properties

$$
T(x+w) = T(x) + T(w)
$$

$$
T(\alpha x) = \alpha T(x).
$$

Definition 4.9. A linear functional is bounded iff its Range is bounded.

Theorem 4.10. Let F be a bounded linear function on L^p, $1 \leq p < \infty$. Then there is a function g in L^q such that

$$
F(f) = \int fg,\ f \in L_p \text{ is arbitrary.}
$$

Proof. Let F be a bounded linear functional on $L^p, 1 \leq p < \infty$. We put

$$\chi_{s(x)} = \begin{cases} 1 & \text{for } 0 \leq x < s \\ 0 & \text{for } s \leq x \leq 1 \end{cases}$$

and show that

$$\Phi(s) = F(\chi_{s(x)})$$

is absolutely continuous For this purpose, let $\{(s_i, t_i)\}$ be any finite collection of non-overlapping subintervals of $[0,1]$ of total length less than δ.

Then

$$\begin{aligned}
\sum_{i=1}^{n} |\Phi(t_i) - \Phi(s_i)| &= \sum_{i=1}^{n} \frac{|\Phi(t_i) - \Phi(s_i)|}{[\Phi(t_i) - \Phi(s_i)]}[\Phi(t_i) - \Phi(s_i)] \\
&= \sum_{i=1}^{n} sgn[\Phi(t_i) - \Phi(s_i)][\Phi(t_i) - \Phi(s_i)] \\
&= F\{\sum_{i=1}^{n} sgn[\chi_{t_i}(x) - \chi_{s_i}(x)][\chi_{t_i}(x) - \chi_{s_i}(x)]\}\} \\
&\leq \|F\| \| \sum_{i=1}^{n} sgn[\chi_{t_i}(x) - \chi_{t_s}(x)][\chi_{t_i}(x) - \chi_{s_i}(x)]\}\| \\
&= \|F\|\{\int_0^1 |\sum_{i=1}^{n} sgn[\chi_{t_i}(x) - \chi_{s_i}(x)][\chi_{t_i}(x) - \chi_{s_i}(x)]\ |^p dx\}^{1/p}.
\end{aligned}$$

If we take $\delta = \frac{\in^p}{\|F\|^p}$, then it follows that total variation Φ is less than $\in$ over any finite collection of disjoint intervals of total length less than δ. Thus Φ is absolutely continuous.

Also we know that a function F is absolutely continuous iff it is indefinite integral. Therefore an integrable function g such that

$$\Phi(s) = \int_0^s g$$

Thus

$$(\chi_s) = \int_0^1 g\chi_s \quad \text{where } \chi_s = \begin{cases} 1, & \text{if } x \in s \\ 0, & \text{if } x \notin s \end{cases}$$

Since every step function on $[0,1]$ is [equal except at a finite number of pts to] to a suitable linear combination $\sum c_i \chi s_i$, we must have

$$F(\psi) = \int_0^1 g\psi \tag{*}$$

For each step function ψ by the linearity of F and of the integral.

Let f be any bounded measurable function on $[0,1]$ [hence Lebesgue integrable]. Then it follows that there is a sequence $< \psi_n >$ of step functions which converges almost everywhere to f. Since the sequence $< |f-\psi_n|^p >$ is uniformly bounded and tends to zero almost every where by the bounded convergence theorem [Let $< fn >$ be a sequence of measurable functions defined on a set E os finite measure and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, then $\int_E f = \lim \int_E fn$] implies that $\|f-\psi_n\|_p \to 0$. Since F is bounded and

$$|F(f)-F(\psi_n)| = |F(f-\psi_n)| \leq \|F\|\|f-\psi_n\|_p$$

we must have

$$F(f) = \lim F(\psi_n) \qquad (**)$$

Since $g\psi_n$ is always less than $|g|$ times the uniform bound for the sequence $< \psi_n >$, we have

$$\int fg = \lim \int g\psi_n \qquad (***)$$

by the Lebesgue convergence theorem (Let g be integrable over E and let $< fn >$ be a sequence of measurable functions such that $|f_n| \leq g$ on E and for almost all x in E we have $f(x) \lim f_n(x)$
Then

$$\int_E f = \lim \int_E f_n.$$

Consequently, we must have

$$\int fg = F(f) \quad \text{using } (*), (**), (***)$$

for each **bounded measurable function** f**.** Since

$$|F(f)| \leq \|F\|\|f\|_p,$$

we have g in L_q and $\|g\|_p \leq \|F\|$ by the Lemma which states that "Let g be an integrable function on $[0,1]$ and suppose that there is a constant M such that $|\int fg| \leq M\|f\|_p$ for all bounded measurable function f. then g is in L_q and $\|g\|_q \leq M$" thus we have only to show that $F(f) = \int fg$ for each f in L^p. Let f be an arbitrary function in L^p. Then there is for each $\in > 0$, a step function ψ such that $\|f-\psi\|_p < \in$. Since is bounded, we have

$$F(\psi) = \int \psi g$$

Hence

$$\begin{aligned} |F(f) - \int fg| &= |F(f) - F(\psi) + \int \psi g - \int fg| \\ &\leq |F(f - \psi)| + |\int (\psi - f)g| \\ &\leq \|F\|\|f - \psi\|_p + \|g\|_q\|f - \psi\|_p \\ &< [\|F\| + \|g\|_q] \in . \end{aligned}$$

Since $\in$ is an arbitrary number, we must have

$$F(f) = \int fg$$

Riesz-Representation theorem for bounded linear functional on $C[a,b]$.

Theorem 4.11. Let $F \in C^*[a,b]$. Then there exists a function $g \in BV[a,b]$ [bounded variation] such that for all $F \in C[a,b]$.

$$F(f) = \int_a^b f(t)dg(t)$$

Such that

$$\|F\| = V(g)$$

Where $V(g)$ denotes the total variation of $g(t)$.

Proof. If we view $C[a,b]$, as a subspace of $B[a,b]$, by Hahn-Banach theorem, there exists a bounded linear functional F_0 defined on all of $B[a,b]$, defined extending F and such that $\|F_0\| = \|F\|$. Define the characteristic function

$$\chi_t(x) = \begin{cases} 1 & \text{for } a \leq x < s \\ 0 & \text{for } s \leq x \leq b \end{cases}$$

Obviously, for each such t,

$$\chi_t(x) \in B[a,b]$$

with F_0 the extension of F, we now define a function $g(t)$ by

$$F_0(\chi_t(x)) = g(t).$$

We partition the interval $[a,b]$ into

$$a = t_0 < t_1 < \ldots < t_n = b$$

and consider the sum

$$\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})|.$$

Putting

$$\in_i = sgn[g(t_i) - g(t_{i-1})] = \frac{|g(t_i) - g(t_{i-1})|}{[g(t_i) - g(t_{i-1})]}$$

we obtain

$$\begin{aligned}\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| &= \in_i [g(t_i) - g(t_{i-1})] \\ &= \sum_{i=1}^{n} \in_i [F_0(\chi_{t_i}) - F_0(\chi_{t_{i-1}})] \\ &= F_0[\sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}})]\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \leq \|F_0\| \| \sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}}) \|$$

because

$$\|F_0\| = \|F\| \text{ and } \| \sum_{i=1}^{n} \in_i (\chi_{t_i} - \chi_{t_{i-1}}) \| = 1$$

Hence

$$| \sum_{i=1}^{n} |g(t_i) - g(t_{i-1})| \leq \|F\|$$

that is $g(t)$ is of bounded variation.
Also it follows that

$$V(g) \leq \|f\| \tag{1}$$

Suppose now that $f \in C[a,b]$ and define

$$Z_n(t) = \sum_{i=1}^{n} f(t_i)[\chi_{t_i}(x) - \chi_{t_{i-1}}(x)]$$

Where the sequence $< Z_n(t) >$ converges strongly to $f(t)$ i.e. $\|Z_n - f\| \to 0$.
Then the equality

$$F_0(Z_n) = \sum_{i=1}^{n} f(t_i)[g(t_i) - g(t_{i-1})]$$

Implies that

$$\begin{aligned}\lim_{n\to\infty} F(Z_n) &= \lim_{n} \sum_{i=1}^{n} f(t_i)[g(t_i) - g(t_{i-1})] \\ &= \int_a^b f(t)dg(t)\end{aligned}$$

by the definition of Riemann-Stieltjes integral. Since the sequence $< Z_n(t) >$ converges strongly to $f(t)$ i.e. $\|Z_n - f\| \to 0$ and F_0 is a **bounded** (or continuous) linear functional and therefore cont, this implies that

$$F_n(Z_n) \to F_0(f)$$

Therefore

$$F_0(f) = \int_a^b f(t)dg(t).$$

Now since f was an arbitrary continuous function on $[a,b]$ and F_0 must agree with F on $C[a,b]$, we can write

$$F(f) = \int_a^b f(t)dg(t) \quad \text{for any } f \in C[a,b] \tag{2}$$

From (2), we have

$$\begin{aligned}|F(f)| &= |\int_a^b f(t)\,dg(t)| \\ &\le \max_{t\in[a,b]} |f(t)|.V(g). \\ &= \|f\|V(g) \\ &= \|f\|V(g) \quad \text{for all } \ f \in C[a,b]\end{aligned}$$

Taking sup $\|f\| \le 1$, we have

$$\|f\| \ge V(g) \tag{3}$$

From (1) and (3), it follows that

$$\|f\| = V(g).$$

CHAPTER 5

Second Conjugate Spaces

We know that the conjugate space N^* of a normed linear space N is itself a normed linear space. As R and C are normed linear spaces, we can form the conjugate space $(N^*)^*$ of N^* and denote this by N^{**} and call it the second conjugate or dual space of N. The importance of N^{**} lies in the fact that each vector x in N give rise to a functional F_x in N^{**} and that there exists an isometric isomorphism of N into N^{**} called the natural imbedding of N into N^{**}. The following definition will be required to establish natural imbedding of N in N^{**}.

Definition 5.1. Let N and N' be normed linear spaces. Then a one to one linear transformation $T : N \to N'$ is called isometric isomorphism of N into $N^{'}$ if $\|Tx\| = \|x\|$ for every $x \in N$. Further if there exists an, isometric isomorphism of N onto N', then N is said to isometrically isomorphic to N'.

We now show that to each vector $x \in N$, there is a functional F_x in N^{**}.

Hence we prove the following result.

Theorem 5.2. Let N be an arbitrary normed linear space. Then for each vector $x \in N$, the scalar valued function F_x defined by

$$F_x(f) = f(x) \quad \forall\, f \in N^*$$

is a continuous linear functional in N^{**} and the mapping $x \to F_x$ is an isometric isomorphism of N into N^{**}.

Proof. Let N be an arbitrary normed linear space. Let x be a vector in N, consider the scalar valued function F_x defined by

$$F_x(f) = f(x) \quad \forall\, f \in N^*$$

We assert that F_x **is linear**. In fact

$$\begin{aligned} F_x(\alpha f + \beta g) &= (\alpha f + \beta g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha F_x(f) + \beta F_x(g) \end{aligned}$$

Now computing the norm of F_x, we have

$$\begin{aligned}
\|F_x\| &= \sup\{|F_x(f)|; \|f\| \leq 1\} \\
&= \sup\{|F(x)|; \|f\| \leq 1\} \\
&\leq \sup\{\|f\|\ \|x\|; \|f\| \leq 1\} \\
&\leq \|x\| \qquad (1)
\end{aligned}$$

Therefore F_x **is bounded** and a continuous linear functional on N^*. [F_x is called the functional on N^* induced by the vector x and is referred to as induced functional]. Now define a mapping $\phi : N \to N^{**}$ by

$$\phi(x) = F_x \quad \forall\, x \in N.$$

Clearly ϕ is one to one, since

$$\begin{aligned}
& \phi(x) = \phi(y) \Rightarrow F_x = F_y \\
\Rightarrow \quad & F_x(f) = F_y(f) \quad \forall\, f \in N^* \\
\Rightarrow \quad & F(x) = f(y) \\
\Rightarrow \quad & f(x-y) = 0 \Rightarrow x - y = 0 \Rightarrow x = y.
\end{aligned}$$

Let $x, y \in N$, then for all scalars α and β,

$$\phi(\alpha x + \beta y) = F_{\alpha x + \beta y}$$

If $f \in N^*$ then

$$\begin{aligned}
F_{\alpha x+\beta y}(f) &= f(\alpha x + \beta y) \\
&= \alpha f(x) + \beta f(y) \\
&= \alpha F_x(f) + \beta (F_y(f) \\
&= (\alpha F_x + \beta F_y)(f) \\
&= \alpha F_x + BF_y
\end{aligned}$$

Thus

$$F_{\alpha x+\beta y} = \alpha F_x + \beta F_y$$

and hence

$$\Rightarrow \quad \phi(\alpha x + \beta y) = \alpha F_x + \beta F_y + \alpha\phi(x) + \beta\phi(y)$$

which shows that ϕ is linear

Moreover by (1)

$$\|\phi(x)\| = \|F_x\| \leq \|x\| \tag{2}$$

Also we know that if x is a non-zero vector in N, then there exists a functional f_0 in N^* such that $f_0(x) = \|x\|$ and $\|f_0\| = 1$. So

$$\begin{aligned}
& \|x\| = f_0(x) \\
& \qquad \leq \sup\{|f_0(x)|; f_0 \in N^* \text{ and } \|f_0\| = 1\} \\
& \qquad = \sup\{|F_x(f_0)|; \|f_0\| = 1\} \\
\Rightarrow \quad & \qquad = \|\phi(x)\| \qquad (3) \\
& \|x\| \leq \|\phi(x)\|
\end{aligned}$$

Thus from (2) and (3)

$$\begin{aligned}
& \|\phi(x)\| = \|x\| \quad \forall\, x \in N. \\
\Rightarrow \quad & \phi \text{ is an isometry.}
\end{aligned}$$

It follows therefore that $x \to F_x$ is an isometric isomorphism of N into N^{**}.

Remark. This isometric isomorphism is called the **natural imbedding** of N into N^{**}, for we may regard N as a part' of N^{**} of without altering any of its structure as a normed linear space and we write

$$N \subset N^{**}.$$

Reflexive Spaces

Definition 5.3. A normed linear space N is said to be reflexive if $N = N^{**}$. The space l_p and l_q for $1 < p < \infty$ are reflexive since $l_p^* = l_q \Rightarrow l_p^{**} = l_q^* = l_p$.

Remark. Every reflexive space is a Banach space since N^{**} is a complete space. But a Banach space may be non-reflexive space for $C[0,1]$ is a Banach space but it is not reflexive.

Example.

$$(l_p^n)^* = l_p^n$$

$$(l_p^n)^* = l_\infty^n, \quad (l_p^n)^* = l_1^n$$

Where

$$l_p^n = \{x = (x_1, x_2, \ldots, x_n) \|x\| = (\sum_1^n |x_i|^p)^{1/p}\}$$

$$l_p^n = \{x = (x_i)_{i-1}^n ; \|x\| = \sum_1^n |x_i|\}$$

$$l_\infty^n = \{x = (x_i)_{i-1}^n ; \|x\| = \max_{1<i<n} |x_i|\}$$

Solution. Let L be the linear space of n tuples $x = (x_1, x_2, \ldots, x_n)$.
If $\{e_1, e_2, \ldots, e_n\}$ is a natural basis of L. Then

$$x = x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$$

If f is any linear functional on L i.e. A scalar valued linear function

$$\begin{aligned} f(x) &= (x_1 e_1 + \ldots + x_n e_n) \\ &= f(x_1 e_1) + \ldots + (x_n e_n) \\ &= x_1 f(e_1) + \ldots + x_n f(e_n) \end{aligned}$$

where x_i's are scalars.
Put $f(e_1) = y_1 \ldots, f(e_n) = y_n$, then $(y_1, \ldots, y_n)$ is an n-tuples of scalars. Thus

$$f(x) = \sum_{i=1}^n x_i y_i \quad \forall\, x = (x_i)_1^n \in L.$$

is a linear functional since

$$\begin{aligned} f(x + x') &= \sum_{i=1}^n (x_i + x_i') y_i \\ &= \sum_{i=1}^n (x_i y_i + x_i' y_i) \\ &= \sum_{i=1}^n x_i y_i + \sum_{i=1}^n x_i' y_i \\ &= f(x) + f(x') \end{aligned}$$

Similarly $f(\alpha x) = \sum_{i=1}^n \alpha x_i y_i = \alpha \sum_{i=1}^n x_i y_i = \alpha f(x)\ \forall\ \alpha$ scalar.
Thus we have a $1-1$, onto mapping defined by

$$y = (y_1, y_2, \ldots, y_n) \to F$$

where $f \in L^*, y \in L$. Thus algebraically $L' = L$.
By defining a suitable norm, say the norm

$$\|x\| = (\sum_{i=1}^{n} |x_i^p|)^{1/p}$$

on L to make it l_p^n space, the L' space of all continuous functionals is equal to $(1_p^n)^*$, where the norm of f is given by

$$\|f\| = \inf\{k; k \geq 0 \text{ and } |f(x)| \leq k\|x\|\} \Rightarrow x \in l_p^n.$$

It is sufficient to show that what norm of $y = (y_1, y_2, \ldots, y_n)$ makes the mapping $y \Leftrightarrow f$ an isometric isomorphism.

Case I: when $1 < p < \infty$

Then we can show that $(l_p^n)^* = l_q^n$

$$\|x\| = (\sum_{1}^{n} |x_i^p|)^{1/p} \ \ \forall\, x \in l_p^n.$$

If f is continuous linear functional

$$\begin{aligned} |f(x)| &= |\sum_{i=1}^{n} x_i y_i| \\ &\leq \sum_{1}^{n} |x_i y_i| \\ &\leq (\sum_{i=1}^{n} |x_i|^p)^{1/p} (\sum_{i=1}^{n} |y_i|^q)^{1/q} \end{aligned}$$

[By using Holder's inequality]

$$|f(x)| \leq (\sum_{i=1}^{n} |y_i|^q)^{1/q} \|x\|$$

Thus we have

$$\|f\| \leq (\sum_{i=1}^{n} |y_i|^q)^{1/q}$$

since

$$|f(x)| \leq \|f\|\, \|x|$$

and

$$\|f\| = \inf(\sum_{i=1}^{n} |y_i|^q)^{1/q}$$

For the other inequality consider the vector x defined by

$$x_i = 0 \quad \text{if} \quad y_i = 0$$

and $x_i = \frac{|y_i|^q}{y_i}$ otherwise.

$$\begin{aligned}
f(x) &= \sum_{i=1}^{n} x_i y_i = \sum_{i=1}^{n} |y_i|^q \\
\frac{|f(x)|}{\|x\|} &= \frac{\sum_{i=1}^{n} |y_i|^q}{(\sum_{i=1}^{n} |x_i|^p)^{1/p}} \\
&= \frac{\sum_{i=1}^{n} |y_i|^q}{(\sum_{i=1}^{n} |y_i|^{p(q-1)})^{1/p}} \text{ since } |y_i|^{q-1} = |x_i| \\
&= \frac{\sum_{i=1}^{n} |y_i|^q}{(\sum_{i=1}^{n} |y_i|^q)^{1/p}} \\
&= (\sum_{i=1}^{n} |y_i|^q)^{1-\frac{1}{p}} = (\sum_{i=1}^{n} |y_i|^q)^{1/q} \\
\Rightarrow \qquad |f(x)| &= (\sum_{i=1}^{n} |y_i|^q)^{1/q} \|x\| \leq \|f\| \|x\|
\end{aligned}$$

So for particular choice of x, we have

$$\Rightarrow \qquad |f(x)| = (\sum_{i=1}^{n} |y_i|^q)^{1/q} \|x\| \leq \|f\| \, \|x\|$$

$$\Rightarrow \qquad (\sum_{i=1}^{n} |y_i|^q)^{1/q} \leq \|f\|$$

Thus necessarily, we have

$$\|f = \|(\sum_{i=1}^{n} |y_i|^q)^{1/q} \Rightarrow f \in l_q^n$$

So $x \in l_q^n \Rightarrow f \in l_q^n$.

Case 2: When $p = 1$, $(l_q^n)^* = l_\infty^n$.

Here we have

$$\|x\| = \sum_{i=1}^{n} |x_i| \text{ where } x \in l_1^n.$$

It follows that

$$|f(x)| = |\sum_{i=1}^{n} x_i y_i| \leq \sum_{i=1}^{n} |x_i y_i|$$

$$= \sum_{i=1}^{n} |x_i||y_i|$$
$$\leq \max_{1<i<n} |y_i| \sum_{i=1}^{n} |x_i| \quad \forall\, x = (x_1, \dots x_n) \in l_1^n.$$

Since we know

$$|f(x)| \leq \|f\| \|x\|$$

we see that $\|f\| \leq \max\limits_{1<i<n} |y_i|$.

Now $\max\limits_{1<i<n} |y_i| = |y_k|$ say for some, $k, 1 \leq k \leq n$.

Choose an $x = (x_1, \dots, x_n)$ such that

$$x_i = 0 \ \text{ if } i \neq k$$
$$= \frac{|y_k|}{y_k}, \text{otherwise}$$

Note that $f \neq 0$, then $\exists y_i \neq 0$ such that $y_k \neq 0$.

Thus $|f(x)| = |\sum_1^n x_i y_i| = \frac{|y_k|.y_k}{y_k} = |y_k|$ by definition of x.

$$\|f\| = \sup_{\|x\|=1} |f(x)| \geq |y_k|$$

since $(0, 0, \dots, \frac{|y_k|}{y_k}, \dots)$ has norm 1

$$\Rightarrow \quad \|f\| \leq \max_{1<i<n} |y_i|$$

So we have $(l_1^n)^* = l_\infty^n$.

Case 3: $(l_1^n)^* = l_\infty^n$.

where $\|x\| = \max\limits_{1<i<n} |x_i|$

we have $f(x) = \sum_{i=1}^n x_i y_i$

$$|f(x)| = |\sum_{i=1}^{n} x_i y_i| \leq \sum_{i=1}^{n} |x_i||y_i|$$
$$= \max_{1<i<n} |x_i| \sum_{i=1}^{n} |y_i|$$

Since $|f(x)| \leq \|f\| \|x\|$

$$\Rightarrow \qquad \|f\| \leq \sum_{i=1}^{n} |y_i|$$

consider the vector x defined by

$$x_i = 0 \text{ if } y_i = 0.$$
$$x_i = \frac{|y_i|}{y_i} \text{ otherwise.}$$

we have

$$|f(x)| = \sum_{i=1}^{n} \frac{|y_i| \times y_i}{y_i} = \sum_{i=1}^{n} |y_i|$$

$$\Rightarrow \qquad \frac{|f(x)|}{\|x\|} = \frac{\sum_{i=1}^{n} |y_i|}{\max\limits_{1<i<n} \{|x_i|\}} = \frac{\sum_{i=1}^{n} |y_i|}{\max\limits_{1<i<n} \{|\frac{|y_i|}{y_i}|\}}$$

$$= \frac{\sum_{i=1}^{n} |y_i|}{\max\limits_{1<i<n} \frac{|y_i|}{y_i}} \sum_{i=1}^{n} |y_i|$$

$$\Rightarrow \qquad |f(x)| = \sum_{i=1}^{n} |y_i| \|x\| \leq \|f\| \|x\|$$

$$\Rightarrow \qquad \sum_{i=1}^{n} |y_i| \leq \|f\|$$

Thus

$$\|f\| = \sum_{i=1}^{n} |y_i| \text{ where } f \in l_1^n$$

Thus $(l_1^n)^* = l_\infty^n$.

Remark. A normed linear space may be complete without being reflexive as we will see

$$(C_0)^* = l_1$$

Where C_0 {space of all convergent sequences converges to zero } and

$$(C_0)^* = l_1^* = l_\infty$$

Thus C_0 is not a reflexive. But C_0 is complete space.

Theorem 5.4. $C[0,1]$ is not regular [reflexive]

Proof. Here $C[0,1]$ denotes the set of all real continuous functions x=x(t) on [0,1] and

$$\|x\| = (\int_0^1 |x(t)|^2 dt)^{1/2}$$

Note that $C[0,1]$ is not a Banach space under this norm.

Assume that $C[0,1]$ is regular. An arbitrary linear functional $F(f)$ defined on the space V of all functions of bounded variation. Then must have the form $F_x\ (f) = f(x)$ for suitably chosen $x \in C[0,1]$. Recalling the general form of functional $C[0,1]$, we can write for an arbitrary $F(f)$,

$$F_x(f) = F(x) = \int_0^1 x(t) df(t) \tag{1}$$

where $F(t)$ denotes the function of bounded variation associated with the functional $f(x) \in C[0,1]$. The functional

$$F_{x_0}(f) = f(t_0+0) - f(t_0-0) \tag{*}$$

assigns to every function $f(t)$ of bounded variation, it jump at the point t_0.

Obviously, $F_{x_0}(f)$ is additive and

$$\begin{aligned} |F_{x_0}(f)| &= |f(t_0+0) - f(t_0-0)| \\ &\leq \underset{0}{\overset{1}{var}}(f) = \|f\| \end{aligned}$$

implies the boundedness of $F_{x_0}(f)$ and the fact that norm of $F_{x_0}(f)$ can not be greater than 1. Also $F_{x_0}(f) \neq 0$ that is to say it is sufficient to consider $F_{x_0}(f_1)$ with

$$f_1(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_0 \\ t & \text{for } t_0 \leq t < 1 \end{cases}.$$

Because of (1), a continuous function $x_0(t)$ can be found such that

$$F_{x_0}(f) = \int_0^1 x_0(t) df(t) \tag{2}$$

By (*) we have

$$F_{x_0}(f_0) = 0$$

for $F_0(t) = \int_0^1 x_0(t)df$

because $f_0(t)$ is continuous on $[0,1]$. But on the other hand

$$F_{x_0}(f_0) = \int_0^1 x_0(t)df_0(t) = \int_0^1 x_0(t)dt > 0$$

because $x_0(t) \neq 0$.. This is a contradiction. Therefore $C[0,1]$ can not be regular (reflexive)

Banach-Steinhaus or Uniform Boundedness Principle

The following theorem i.e. Uniform Boundedness Principle enables us to determine whether the norms of a given collection of bounded linear transformations $\{T_i\}$ have a finite least upper bound or equivalently if there is some uniform bound for the set $(\|T_i\|)$. So we prove the following results:

Theorem 5.5. Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector in B, then $(\|T_i\|)$ is a bounded set of numbers that is $\{T_i\}$ is bounded as a subset of $\beta(B,N)$.

Proof. For each positive integer n, let

$$F_n = \{x; x \in B \,\text{and}\, \|T_i(x)\| \leq n \,\text{for all}\, i\}$$

we claim that F_n is a closed subset of B. To show this let y be a limit point of F_n. Then there exists $x \in F_n$ such that $x \neq y$ and $\|x-y\| < \delta$. But since T_i are continuous, we have

$$\|T_i(x) - T_i(y)\| < \in \text{ whenever } \|x-y\| < \delta.$$

Now $T_i(y) = T_i(y-x+x)$

and so

$$\begin{aligned}
\|T_i(y)\| &= \|T_i(y-x) + T_i(y)\| \\
&\leq \|T_i(y-x)\| + \|T_i(x)\| \\
&= \|T_i(y) - T_i(x)\| + \|T_i(x)\| \\
&< \in + n \text{ whenever } \|x-y\| < \delta \\
&\leq n.
\end{aligned}$$

Hence $y \in F_n$. Thus F_n is closed. Also by our assumption

$$B = \bigcup_{n-1}^{\infty} F_n$$

Since B is complete, using Baire's Theorem, we see that one of the F_n, say F_{n_i} has non-empty interior and thus contains a closed sphere S_0 with centre x_0 and radius $r_0 > 0$. Therefore each vector in every set $T_i(S_0)$ has norm less than or equal to n_0, that is $\|T_i(S_0)\| < n_0$.

Clearly $S_0 - x_0$ is the closed sphere with radius r_0 centred on the origin and so $\frac{S_0 - x_0}{r_0}$ is the closed unit sphere S. Since x_0 is in S_0, we have

$$\begin{aligned} \|T_i(S_0 + x_0)\| &= \|T_i(S_0) + T_i(x_0)\| \\ &\leq \|T_i(S_0)\| + \|T_i(x_0)\| \\ &\leq n_0 + n_0 = 2n_0. \end{aligned}$$

This yields

$$\|T_i(S)\| = \|T_i \frac{S_0 - x_0}{r_0}\| \leq \frac{2n_0}{r_0}$$

and therefore

$$\begin{aligned} \|T_i\| &= \sup\{\|T_i(S)\|; \|S\| \leq 1\} \\ &\leq \sup\{\frac{2n_0}{r_0}\} \\ &= \frac{2n_0}{r_0} \text{ for every } i. \end{aligned}$$

which completes the proof of the theorem.

Consequences of Uniform Boundedness Principle

We prove some consequence of **Banach-Steinhaus Theorem** (Uniform Boundedness Principle) having several applications in analysis.

Theorem 5.6. A non empty subset X of a normed linear space N is bounded if and only if $f(X)$, is a bounded set of numbers for each f in N^*.

Proof. Since $|f(x)| \leq \|f\|.\|x\|$, it follows that if X is bounded, then $f(X)$ is also bounded for each f.

To prove the converse, we write $X = \{x_i\}$. We now use natural imbedding to map X to the subset $\{F_{x_i}\}$ of N^{**}. The assumption that $f(X) = \{f(x_i)\}$ is bounded for each f implies that for $\{f(x_i)\}$ is bounded for each f. Moreover since N^* is complete, uniform boundedness theorem shows that $\{F_{x_i}(f)\}$ is a bounded subset of. Since natural imbedding preserves norms, therefore X is evidently a bounded subset of N. This completes the proof of the theorem.

Theorem 3. Let X be a Banach space and Y a normed linear space. Let $\{T_n\}$ be a sequence of terms from $\beta(X,Y)$ conversing strongly to T. Then there exists a positive constant M such that $\|T_n\| < M$ for all n.

Proof. Since $T_n \xrightarrow{S} T$, then

$$\lim_{n\to\infty} T_n x = Tx \text{ for all } x.$$

This implies that

$$\sup_n \|T_n(x)\| < \infty \text{ for all } x.$$

Now using uniform boundedness principle, we must have

$$\sup_n \|T_n\| < \infty.$$

and therefore the theorem is proved.

Definition 5.7. Let $\{T_n\}$ be a sequence of linear transformation from $\beta(X,Y)$.

Then $\{T_n\}$ is said to be a strong Cauchy sequence if the sequence $\{T_n(x)\}$ is a Cauchy sequence for all $x \in X$.

Further a space $\beta(X,Y)$ is said to be complete in the strong sense if every strong Cauchy sequence in $\beta(X,Y)$ converges strongly to some member of the space.

We now prove the following:

Theorem 5.8. If the spaces X and Y are Banach spaces, then $\beta(X,Y)$ is complete in the strong sense.

Proof. Let $< T_n >$ be a strong Cauchy sequence in $\beta(X,Y)$. We must show that there is some element T of $\beta(X,Y)$ to which $< T_n >$ converges strongly.

Since $< T_n >$ is a strong Cauchy sequence, it follows by definition that for any $x \in X, < T_n x >$ is a Cauchy sequence of elements of Y. Since Y is a Banach space, the limit of this sequence must exist in Y. Thus for any $x \in X$, the function

$$Tx = \lim_n T_n x \tag{1}$$

Is defined. Clearly, T is linear transformation and (1) is equivalent to saying that

$$T_n \to T.$$

It remains to show that T is a bounded linear transformation. Since X is a Banach space and $< T_n >$ converges strongly to T, theorem 3 implies that $\|T_n\| < M$, for all n and some positive constant M.

Since for any $x \in X$, we can say

$$\|T_n x\| \leq \|T_n\|.\|x\|$$

this implies that

$$\|T_n(x)\| \leq M.\|x\|$$

for any x and every n. Since it is true for every n, it must also be true in the limit. Thus

$$\lim_n \|T_n(x)\| \leq M.\|x\|.$$

Since norm is continuous, we have

$$\|\lim_n T_n x\| \leq M.\|x\|$$

or

$$\|Tx\| \leq M.\|x\|$$

for every x. Hence T is bounded. Thus we have shown that every strong Cauchy sequence in $\beta(X,Y)$ converges strongly to some element T of $\beta(X,Y)$. Hence $\beta(X,Y)$ is complete in the strong sense and the proof is complete.

We now define what is meant by a week Cauchy sequence of elements of the normed linear space X.

Definition 5.9. The sequence of element $\{T_n\}$ of the normed linear space x is said to be a weak Cauchy sequence if $< f(x_n) >$ is a Cauchy sequence of elements for all $f \in X^*$, the conjugate space of X.

Theorem 5.10. In a normed linear space X, every Cauchy sequence is bounded.

Proof. Let $< x_n >$ be a weak Cauchy sequence of elements of a normed linear space X. This means that $< f(x_n) >$ is a Cauchy sequence for all $f \in X$. We recall the natural imbedding

$$\phi : X \to X^{**}$$

$$x \to F_x$$

where $F_x(f) = f(x)$ for all $x \in X$ and $f \in X^*.\phi$ is a bounded linear functional satisfying

$$\|\phi(x)\| = \|x\| \text{ for all } x \in X.$$

Since $< f(x_n) >$ is a Cauchy sequence of complex numbers, for any $f \in X^*$, we have

$$\sup_n |F_{x_n}(f)| = \sup_n |F_{x_n}(f)| < \infty \tag{1}$$

But X^* is a Banach space. Therefore by Uniform Bounded Principle (1) yields

$$\sup_n |F_{x_n}(f)| < \infty$$

Since

$$\|F_{x_n}\| = \|\phi(x_n)\| = \|x_n\|$$

therefore $\sup_n \|x_n\| < \infty$.

Hence the weak Cauchy sequence $\{x_n\}$ is bounded. This completes the proof.

Theorem 5.11. In a normed linear space X, if the sequence $< x_n >$ converges weakly to x, that is $x_n \to x$, then there exists some positive constant m such that $\|x_n\| < m$ for all n.

Proof. We note that if

$$x_n \xrightarrow{W} x.$$

then certainly $< x_n >$ is a weak Cauchy sequence, Hence by Theorem 5, $\{x_n\}$ is bounded, that is $\|x_n\| \leq m$ for constant m and the proof is complete.

After having introduced the definition of weak Cauchy sequence, we give the following definition of weak completeness of a space.

Definition 5.12. A normed linear space X is said to be weakly complete if every Cauchy sequence of elements of X converges weak to some other member of X.

Our next theorem shows that any reflexive space is weakly complete.

Theorem 5.13. If the normed linear space X is reflexive, then it is also weakly complete.

Proof. Suppose $< x_n >$ is a weak Cauchy sequence of elements of X. this means that $< f(x_n) >$ is a Cauchy sequence for all $f \in X^*$. Now we consider natural imbedding

$$\begin{aligned} \phi :& X \to X^{**} \\ & x \to F_x \end{aligned}$$

This mapping implies that $< F_{x_i}(f) >$ is a cauchy sequence of scalars for all $f \in X^*$.

Since the underlying field is either real or complex (each of which is complete metric space). This implies that the functional y defined on X^{**} by

$$y(f) = \lim_n F_{x_n}(f)$$

exist for every $f \in X^*$. It can be verified that y is linear. We shall now show that y is a bounded linear functional. Since $\|F_{x_n}\| = \|F_x\|$ and $< x_n >$ is a Cauchy sequence, it follows by Theorem 5, that there is some positive number M such that

$$\|x_n\| \leq M.$$

for all n, this implies that

$$\begin{aligned} |F_{x_n}(f)| = |f(x_n)| &\leq \|f\|.\|x_n\| \\ &\leq M.\|f\| \end{aligned}$$

for any $f \in X^*$ and all n. Hence it is true in the limit that is

$$\begin{aligned} & \lim |F_{x_n}(f)| \leq M\|f\| \\ \Rightarrow \quad & |\lim F_{x_n}(f)| \leq M\|f\| \\ \text{or} \quad & |y(f)| \leq M\|f\| \quad \text{using (1)} \end{aligned}$$

for all $f \in X^*$ and all n.

This however implies that y is a bounded linear functional or that $y \in X^{**}$.

Since X is reflexive there must be some $x \in X$ that we can identify with y that is, there must be some $x \in X$ such that $y = F_x$.

Hence for any $f \in X$, we can say

$$\begin{aligned} \lim_n f(x_n) &= \lim_n F_{x_n}(f) \\ &= y(f) \\ &= F_x(f) \end{aligned}$$

$$= f(x)$$

Since this holds for any $f \in X^*$, we have

$$x_n \xrightarrow{w} x.$$

Thus we have shown that each weak Cauchy sequence of elements of X converges weakly to some other member of X. Hence X is weakly complete and the proof of the theorem is complete.

CHAPTER 6

Open Mapping and Closed Graph Thorems

First we present some definitions which will be required in the sequel. The validity of many important therorems of analysis depends on the completeness of the systems with which they deal. Baire's theorem about complete metric spaces is the basic tool in this area. In order to emphasize the role played by the concept of category, some theorems of this chapter are stated in a little more generally than is usually needed.

Definition 6.1. If $T : V \to W$ is a linear transformation, then the set N of all vectors $x \in V$ such that $Tx = 0$ is called the null space (or kernel) of T. Thus

$$N = \{x \in V; Tx = 0\}$$

Also $Tx_1 = Tx_2 \Leftrightarrow T(x_1 - x_2) = 0 \Leftrightarrow x_1 = x_2 \in N$ and that if $x \in N$, then $Tx = 0$ so that if T is injective (one to one). Thus we have shown that T is injective if and only if $N = \{0\}$.

Now suppose that X and Y are normed linear spaces and $T : X \to Y$ is a continuous linear mapping. Let $x_0 \in N$ (null space of T) and let $x_n \to x$. Since T is continuous $Tx_n \to Tx$ thus $Tx = \lim_{n\to\infty} Tx_n = 0$. Hence $x \in N$. This proves that if $T : X \to Y$ is continuous, then null space of T is closed.

Definition 6.2. Let X and Y be normed linear spaces. Then a linear mapping $T : X \to Y$ will be called open mapping if it maps open sets into open set.

Definition 6.3. The mapping $T : X \to Y$ where X and Y are normed spaces as will be called a homeomorphism if it is bijective, continuous and open or equivalently $T : X \to Y$ is a homeomorphism if it is bijective and bi-continuous.

Definition 6.4. Let E be a normed linear space. A subset A of E is called nowhere dense in E if $\overline{A}$ has an empty interior. Q is everywhere dense in R while integers are nowhere dense in R. Thus a nowhere dense set is thought of a set which does not cover much of the space.

Baire Category Theorem.
It states that a complete space can not be covered by any sequence of no-where dense sets.

Open mapping Theorem or Interior Mapping Principle

Theorem 6.5. Let B and B be Banach spaces. If T is a continuous linear transformation of B onto B, then T is an open mapping. (Thus if the mapping T is also one to one, then T^{-1} is continuous).

Proof: First of all, we prove a Lemma

Lemma. Let B and B' be Banach spaces. If T is a continuous linear transformation of B onto B', then the image of each open sphere centred on the origin in B contains an open sphere centred on the origin, in B'.

Proof. Let S_r and S_r' be open spheres with radius r centred on the origin in B and B' respectively. Then

$$T(S_r) = T(rS_1) = rT(S_1)$$

So, it is sufficient to show that $T(S_1)$ contains some S_r'.

We first prove that $\overline{T(S_1)}$ contains some S_r'. Since T is onto, we note that

$$B' = \bigcup_{n=1}^{\infty} T(S_n).$$

Being a Banach space, B' is complete and so by Baire's theorem, some $\overline{T(S_{n_0})}$ has an interior point y_0 lying in $T(S_{n_0})$. Since the mapping $y \to y - y_0$ is a homeomorphism of B' onto itself. $\overline{T(S_{n_0})} - y_0$ has the origin as an interior point. Since y_0 is in $T(S_{n_0})$ we have

$$T(S_{n_0}) - y_0 \subseteq T(S_{2n_0})$$

which in turn implies that

$$\overline{T(S_{n_0})} - y_0 = \overline{T(S_{n_0}) - y_0} \subseteq \overline{T(S_{2n_0})}$$

which shows that the origin is an interior point of $\overline{T(S_{2n_0})}$. As we know that multiplication by any non-zero scalar is a homeomorphism of E' onto itself. So

$$\overline{T(S_{2n_0})} = \overline{2n_0 T(S_1)} = 2n_0 \overline{T(S_1)}$$

and hence the origin is also an interior point of $\overline{T(S_1)}$. Thus $S'_\in \subseteq \overline{T(S_1)}$ for some positive number $\in$. We complete the proof by showing that $S'_\in \subseteq \overline{T(S_1)}$ which is equivalent to $S'_{\in/2} \subseteq T(S_1)$.

Let $y \in B'$ be such that $\|y\| < \in$. Since y is in $\overline{T(S_1)}$, there exists a vector x_1 in B such that $\|x_1\| < 1$ and $\|y - y_1\| < \frac{\in}{2}$, where $y_1 = T(x_1)$. Further $S'_{\in/2} \subset \overline{T(S_{1/2})}$ and $\|y - y_1\| < \frac{\in}{2}$, there exists a vector x_2 in B such that $\|x_1\| < \frac{1}{2}$ and $\|(y - y_1) - y_2\| < \frac{\in}{4}$ where $y_2 = T(x_2)$, continuing in this way, we get a sequence $< x_n >$ in B such that $\|x_n\| < \frac{1}{2^{n-1}}$ and

$$\|y - (y_1 + y_2 + \ldots + y_n)\| < \frac{\in}{2^n}$$

where $y_n = T(x_n)$. Let $S_n = x_1 + x_2 + \ldots + x_n$, then

$$\begin{aligned}
\|S_n\| &= \|x_1 + x_2 + \ldots + x_n\| \\
&\leq \|x_1\| + \|x_2\| + \ldots + \|x_n\| \\
&< 1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}} < 2
\end{aligned}$$

Also for $n > m$, we have

$$\begin{aligned}
\|S_n - S_m\| &= \|x_{m+1} + x_{m+2} + \ldots + x_n\| \\
&\leq \|x_{m+1}\| + \|x_{m+2}\| + \ldots + \|x_n\| \\
&< \frac{1}{2^m} + \ldots + \frac{1}{2^{m-1}} + \ldots + \frac{1}{2^{n-1}} \\
&= \frac{\frac{1}{2^m}(1 - \frac{1}{2^{n-m}})}{1 - \frac{1}{2}} \\
&= \frac{1}{2^{m-1}}[1 - \frac{1}{2^{n-m}}] \to 0 \text{ as } m, n \to \infty.
\end{aligned}$$

Hence $\{S_n\}$ is a Cauchy sequence in B and since B is complete, there exists a vector x in B such hat $\lim\limits_{n\to\infty} S_n = x$ and so

$$\|x\| = \|\lim S_n\| = \lim \|S_n\| \leq 2 \leq 3$$

which implies that $x \in S_3$. Now

$$y_1 + y_2 + \ldots + y_n = T(x_1) + T(x_2) + \ldots + T(x_n)$$

since T is continuous, $x = \lim S_n$

$$\begin{aligned}
\Rightarrow \qquad Tx &= \lim_n (TS_n) \\
&= \lim(y_1 + y_2 + \ldots + y_n) \\
\Rightarrow \qquad Tx &= y
\end{aligned}$$

Thus $y = Tx$ where $\|x\| < 3$ so that $y \in T(S_3)$.

Hence we have proved that

$$y \in S'_{\in} \Rightarrow y \in T(S_3) \text{ and so } S'_{\in} \subseteq T(S_3)$$

Proof of Main Theorem: It is sufficient to show that if G, is an open set in B, then $T(G)$ is also open in B'. To show it let $v \in T(G)$ we shall show that y is an interior point of $T(G)$ i.e. there exists an open sphere centered on y and contained in $T(G)$. Let x be a point in G such that $y = Tx$. Since G is open, x is an interior point of G.

Therefore x is the centre of an open sphere written in the form $x = Sr$, contained in G. Hence by the above Lemma, $T(S_r)$ contains some sphere S'_{r_1}. Then $y + S'_{r_1}$ is an open sphere centred on y.

Moreover

$$\begin{aligned} y + S'_{r_1} &\subseteq y + T(S_r) \\ &= T(x) + T(S_r) \\ &= T(x + Sr) \\ &\subseteq T(G) \end{aligned}$$

Hence $y + S'_{r_1}$ is an open sphere centred on y and contained in $T(G)$. Consequently $T(G)$ is open. Hence the result.

Corollary: A one to one continuous linear transformations of one Banach space onto another is a homeomorphism.

Proof: The given hypothesis yields that the linear transformation is bijective and continuous. Further by open mapping theorem, the linear transformation is also open. Hence it is homeomorphism.

Projections on Banach spaces

Definition 6.6. Let L be a vector space. We say that L is the direct sum of its subspace say M and N ; if every element $z \in L$ has a unique representation $z = x + y$ with X in M and y in N. In such a case we write $L = M \oplus N$.

Define a mapping $P : L \to L$ by $P(z) = x$. Then P is a linear transformation, then

(i) $P(z) = z$ if and only if $z \in M$

(ii) $P(z) = 0$ if and only if $z \in M$

(iii) P is idempotent that is $P^2 = P$. Infact

$$P^2(z) = P(P(z)) = P(x) = x = P(z).$$

Such a linear mapping P is called a projection on the linear space L.

Thus if L is the direct sum of its subspaces M and N, then there exists a linear transformation P which is idempotent.

But, however in case of Banach spaces, more is required of a projection than more linearity and idempotence we have

Definition 6.7. A projection on a Banach space is a projection on B in the algebraic sense (linear and idempotent) which is also continuous.

It follows from the above discussion that if B is the direct sum of its subspaces M and N, then there exists a linear transformation P which is idempotent. Further we have

Theorem 6.8. If P is a projection on a Banach space B and if M and N are its range and null space, then M and N are closed linear subspaces of B such that $B = M \oplus N$.

Proof. We are given that P is a projection on a Banach space B and M and N are range and null spaces of. Thus M is linear, continuous and idempotent and

$$M = \text{range of } P = \{P(z); z \in B\}$$
$$N = \text{null space of } P = \{z; P(z) = 0\}$$

Let $z \in B$. Consider

$$z = P(z) + (I - P)z \tag{1}$$

where I denotes the identity transformation on B such that $I(z) = z$ for all $z \in B$.

Clearly $p(z)$ is in M and since P is idempotent, we have

$$\begin{aligned} P\{(I-P)(z)\} &= P\{(I-P)\}(z) \\ &= (P - P^2)(z) \\ &= (P - P^2)(z) = 0(z) = 0 \end{aligned}$$

It follows therefore that $(I - P)(z) \in N$, the null space of P. Therefore equation (1) gives a de composition of z according to the subspaces M and N. This decomposition is unique because if we have another representation as $z = x + y$, $x \in M$, $y \in N$ then

$$P(z) = P(x) = x$$

and

$$\begin{aligned}(I-P)(z) &= I(z)-P(z)\\ &= z-x\\ &= y\end{aligned}$$

Thus $B = M \oplus N$. We know that the null space of a continuous linear transformation is closed. Therefore continuity of P implies that N is closed.

Further, since $M = \{P(z); z \in B\} = \{x; P(x) = x\}$

$$\Rightarrow \qquad M = \{x; (I-P)(x) = 0\}$$

It follows that M is the null space of continuous linear transformation $I-P$ and hence closed. Thus M and N are closed and $B = M \oplus N$. Hence the result.

As an application of open mapping theorem, we have

Theorem 6.9. Let B be a Banach space and let M and N be closed linear subspaces of B such that $B = M \oplus N$. If $z = x+y$ is the unique representation of a vector in B as the sum of vectors in M and N, then the mapping P defined by $P(z) = x$ is a projection on B whose range and null space are M and N.

Proof. Let $P : B \to B$ be defined by $P(z) = x$ for $z = x+y$, $x \in M$, $y \in N$. Then since $P(z) = x$ for $z \in B$, we have M to be the range of P. Also $P(y) = 0$ for $y \in N$. Therefore N is the null space of P.

Further

$$P^2(z) = P(P(z)) = P(x) = x = P(z)$$

Implies that P is idempotent. Hence to prove that P is a projection on B, it only remains to show that P is continuous. Let

$$z = x+y, x \in M, y \in N$$

be unique representation of the elements of the Banach space B. Define a new norm on B by

$$||z||' = ||x|| + ||y||$$

and let B' denote the linear space B equipped with this new norm, then B' is a Banach space and since

$$P(z)|| = ||x|| \leq ||x|| + ||y|| = ||z||'$$

It follows that P in continuous as a mapping of B' into B,. It is therefore sufficient to show that B and B' are homeomorphic. Let T denote the identity mapping of B' onto B. Then

$$\|T(z)\| = \|z\| = \|x+y\| \le \|x\| + \|y\| = \|z\|'.$$

Shows that T is one to one continuous linear transformation of B' onto B.

Open mapping theorem now implies that T is a homeomorphism. Thus B and B' are homeomorphic. Hence $P : B \to B$ is continuous and therefore a projection on B.

Closed Linear Transformations and Closed Graph Theorem

Let X and Y be normed linear spaces. Then the Cartesian product $X \times Y$ of X and Y becomes a normed linear space under the norm defined by

$$\|(x,y)\| = \|x\| + \|y\|$$

Further if X and Y are Banach spaces, then $X \times Y$ is also a Banach space w.r.t. the norm defined above.

Definition 6.10. Let $T : B \times B'$ be a linear transformation of a Banach space into another Banach space B'. Then the collection of ordered pairs.

$$G_T = \{(x,Tx); (x,Tx) \in B \times B'\}$$

is called the graph of T. It can be shown that G_T is a linear subspace of $B \times B'$.

Definition 6.11. Let X and Y be normed linear spaces and let D be a subspace of X. Then the linear transformation $T : D \to Y$ is called closed if $\{x_n\} \in D, \lim\limits_n x_n = x$ and $\lim\limits_n Tx_n = y \in Y$ imply $x \in D$ and $y = Tx$.

As justification for the name given closed transformation in the above definition, we now show that a linear transformation T is closed iff its graph G_T is a closed subspace of $X \times Y$.

Theorem 6.12. A linear transformation is closed iff its graph is a closed subspace.

Proof. Let X and Y be normed linear spaces and let D be a subspace of X.
Suppose first that $T : D \to Y$ is a closed linear transformation. To show that G_T is closed, we must show that any limit point of G_T is actually a member of G_T. Therefore

there must be a sequence of points of $G_T, (x_n, Tx_n), x_n \in D$ converging to (x,y), this is equivalent to

$$\|(x_n, Tx_n)\| - \|x,y\| \to 0$$

or

$$\|(x_n - x, Tx_n - y)\| \to 0$$

or

$$\|x_n - x\| + \|Tx_n - y\| \to 0 \Rightarrow \qquad x_n \to x \text{ and } Tx_n \to y$$

Since T is closed, this implies that $x \in D$ and $y = Tx$.

Therefore we can write that

$$(x,y) = (x,Tx) \in G_T$$

$\Rightarrow$ Every limit pt (x,y) of GT is a member of G_T.

$\Rightarrow$ G_T is closed.

Conversely suppose that G_T is closed, and let $x_n \to x, x_n \in D$, for all n as $Tx_n \to y$.

We must show that $x \in D$ and $y = Tx$. The condition implies that

$$(x_n, Tx_n) \to (x,y) \in \overline{G_T}$$

Since G_T is closed we have

$$G_T = \overline{G_T}$$

and thus we have

$$(x,y) \in G_T$$

But by the definition of G_T, this means that $x \in D$ and $y = Tx$. Hence T is a closed linear transformation. This completes the proof of the theorem. The next things we wish to investigate is when a bounded (continuous) transformation is closed. Infact, we prove.

Theorem 6.13. Let X and Y be normed linear spaces and let D be a closed subspace of X. If $T : D \to Y$ is bounded, then T is closed.

Proof. D is a closed subspace of X and $T : D \to Y$ is bounded. If $< x_n >$ is a convergent sequence of points of D such that $Tx_n \to y$, then D being closed, the limit of the sequence $< x_n >$ must belong to D. On the other hand, the continuity (boundedness) of T implies that $Tx_n \to Tx$. Hence $y = Tx$. (since $Tx_n \to y$). Thus T becomes closed. Hence the result.

An immediate consequence of the theorem is of the following :

Corollary. Suppose T is linear transformation from a normed linear space X into another normed linear space Y. If T is continuous, then T is closed. Also then using Theorem A, G_T is closed.

Proof. We know that the entire space X is always closed, therefore Theorem B applies and the result follows.

Theorem 6.14. Let X and Y be normed linear spaces and let D be a subspace of X. If $T : D \to Y$ is a closed linear transformation, then T^{-1} (if exists) is also a closed linear transformation.

Proof. Since T is closed, its graph.

$$G_T = \{(x, Tx); x \in D\}$$

is closed, let $T(D)$ denote the range of T. Since T^{-1} exists, for any $y \in T(D)$, there is a unique $x \in D$ such that $y = Tx$ or $T^{-1}(y)$. Therefore graph of T can be written as

$$G_T = \{(T^{-1}y; y); y \in T(D)\}$$

Consider now the mapping

$$X \times Y \to Y \times X$$

$$(x, y) \to (y, x)$$

This mapping is isometry, since Isometrics map closed sets into closed sets and the set $\{(T^{-1}y, y), y \in T(D)\}$ is closed. It follows that the set $\{(y, T^{-1}y), y \in T(D)\}$ is also closed. But this last set is just the graph of T^{-1}. Thus we have proved that that graph of T^{-1} is closed or hence T^{-1} is closed by Theorem A.

Theorem 6.15. Let D be a subspace of a normed linear space X and let $T : D \to Y$ be a linear transformation from D into a Banach space Y. If T is closed and bounded, then D is a closed subspace of X.

Proof. It is sufficient to show that any limit point of D is also a member of D.

Hence suppose that x is a limit point of D. This means that there must be some sequence $\{x_n\}$ of points of D such that $x_n \to x$. Consider now

$$\|Tx_n - Tx_m\| \leq \|T\| \ \|x_n - x_m\|$$

Since

$$\|x_n - x_m\| \to 0 \text{ as n,m} \to \infty$$

as every convergent sequence is Cauchy.

It follows that $< Tx_n >$ is a Cauchy sequence in Y. But Y being a Banach space is complete. Therefore there exists $x \in Y$ such that

$$Tx_n \to y.$$

Thus we have $x_n \to x$ and $Tx_n \to Y$.. Now since T is closed. This implies that $x \in D$. Hence D contains all its limit points and hence closed. This completes the proof of the theorem.

We now state and prove Closed Graph Theorem.

Closed Graph Theorem

Theorem 6.16. Let B and B' be Banach spaces and let T : B B be a linear transformation. Then graph of T is closed if and only if T is continuous.

Proof. Suppose first that T is continuous. Then Corollary to Theorem B implies that G_T is closed.

Conversely suppose that G_T is closed. Since B and B' are Banach spaces. It follows that $B \times B'$ is a Banach space. Since closed subsets of a complete metric space must be complete, it follows that G_T (being closed) is Banach space too. Now consider the mapping

$$f : G_T \to B$$

defined by

$$f(x, Tx) = x$$

Clearly f is a linear transformation. We claim further that f is bounded. To prove this, we note that

$$\begin{aligned} \|f(x,Tx)\| &= \|x\| \leq \|x\| + \|Tx\| \\ &= \|(x,Tx)\| \end{aligned}$$

which implies that f is a bounded linear transformation. Further $f(G_T) = B$ and therefore f is onto. We shall show that f is one to one. Also we know that a linear transformation is one-to-one if its kernel (null space) consists of identity element only. Therefore. We need to prove that $(0,0)$ is the only element f maps into zero. Hence, suppose

$$f(x, Tx) = x = 0$$

But $x = 0$ implies that $Tx = 0$ and so

$$(x, Tx) = (0, 0)$$

and hence f is one to one. Thus $f : G_T \to B$ is bijective and therefore f^{-1} exists.

Now G_T and B and Banach spaces and f is a continuous linear transformation and f^{-1} is continuous. To complete the proof we must show that if $x_n \to x$, then $Tx_n \to Tx$. [T is continuous]. Hence suppose that $x_n \to x$.

Since f^{-1} is continuous, we have

$$\begin{aligned} & f^{-1}x_n \to f^{-1}x, \\ \Rightarrow \quad & (x_n, Tx_n) \to (x, Tx) \\ \Rightarrow \quad & (x_n - x, Tx_n - Tx) \to (0, 0) \\ \Rightarrow \quad & Tx_n \to Tx \end{aligned}$$

Thus T is continuous. Hence the result.

Equivalent Norms

Suppose X is a vector space over the scalar field F and suppose that $\|.\|_1$ and $\|.\|_2$ are each norms on X, Then $\|.\|_1$ is said to be equivalent to $\|.\|_2$ written as $\|.\|_1 \sim \|.\|_2$,, if $\exists$ positive numbers a and b such that

$$a\|x\|_1 \leq \|x\|_2 \leq \|x\|_1 \text{ for all } x \in X.$$

This relation is an equivalence relation on the set of all norms over a given space. Further, if two norms are equivalent, then certainly if $< x_n >$ is a Cauchy sequence with respect to $\|.\|_1$ it must also be a cauchy sequence with respect to $\|.\|_2$ and vice-versa.

Let a basis for he finite dimensional space be $[x_1, x_2, \ldots, x_n]$. For any $x \in X$, there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $x = \sum_{i=1}^{n} \alpha_i x_i$.. Now $\|x\|_0 = \max_i |\alpha_i|$ is indeed a norm. This norm is called **Zeroth Norm.** We

Theorem 6.17. On a finite dimensional space, all norms are equivalent.

Proof. We shall show that all norms are equivalent by showing that any norm is equivalent to the particular norm defined above and called the **Zeroth norm.**

Let a basis for the finite dimensional space X is given by $x_1, x_2, \ldots, x_n$. For any $x \in X_1$ there exist unique scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i. \tag{*}$$

Now $\|x\|_0 = \max_i |\alpha_i|$ is indeed a norm.

Now let $\|\|$ be any norm on X. We want to find real numbers $a, b > 0$ such that (1) is satisfied, where $\| \ \|_2$ is replaced b $\|.\|$ and $\|.\|_1$ is replaced by $\| \ \|_{0.}$.

The right hand side of (1) easily satisfies

$$a\|x\|_1 \leq \|x\|_2 \leq b\|x\|_1 \tag{1}$$

since from (*)

$$\begin{aligned}\|x\| = \|\sum_{i=1}^{n} \alpha_i x_i\| &\leq \sum_{i=1}^{n} |\alpha_i| \ \|\alpha_i\| \\ &\leq \max_i |\alpha_i| \sum_{i=1}^{n} \|x_i\| \\ &\leq \|x\|_0 \sum_{i=1}^{n} \|x_i\|\end{aligned}$$

because, since the basis is fixed, we can take as the number b

$$b = \sum_{i=1}^{n} \|x_i\|$$

to get for any $x \in X$,

$$\|x\| \leq b\|x\|_0$$

The left side of (1) does not follow quite as simply. Consider the simple case of a one-dimensional space with basis x_1. Any vector in the space X can be written uniquely as

$$x = \alpha_1 x_1$$

for some $\alpha_1 \in F$. Hence

$$\|x\| = |\alpha| \ \|x_1\|$$

Thus in this case, the number a on the left side of (1) can be taken to be just $\|x_1\|$.

Having verified this, we shall now proceed by induction, suppose the theorem is true for all spaces of dimension less than or equal to $n-1$. We can now say that, if $\dim X = n$, with basis $\{x_1, x_2, \ldots, x_n\}$ and

$$M =< x_1, x_2, \ldots, x_{n-1} >$$

be the subspace spanned by the first $n-1$ basis vectors, then

$$\| \, \| \sim \| \, \|_0$$

in M. Since this is so, if $\{y_n\}$ is a cauchy sequence of elements from M w.r.t. to $\| \, \|$, then $\{y_n\}$ is also a cauchy sequence with respect to $\| \, \|_0$. Consider the ith term of this sequence now :

$$y_i = \alpha_1^{(i)} + \alpha_2^{(i)} x_2 + \ldots + \alpha_{n-1}^{(i)} x_{n-1}$$

By the above

$$\|y_n - y_m\|_0 \to as\, n, m \to \infty \tag{2}$$

Since $\{y_n\}$ is a cauchy sequence.

But $\|y_n - y_m\|_0 = \max_j |\alpha_j^{(n)} - \alpha_j^{(m)}|$ which by (2) implies

$$|\alpha_j^{(n)} - \alpha_j^{(m)}| \to 0, \; as\, n, m \to \infty \tag{3}$$

for $j = 1, 2, \ldots n-1$. Since $F = R$ or C, and each is complete and (3) states that if the $\{\alpha_j^{(m)}\}$ is a cauchy sequence, there must exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ such that

$$\alpha_j^{(m)} \to \alpha_j (j = 1, 2, \ldots, n-1)$$

In view of this, it is clear that

$$y_m \to y = \sum_{j=1}^{n} \alpha_i x_i$$

with respect to the zeroth norm. Further

$$y_m \xrightarrow{\| \|_0} y \Rightarrow y_m \xrightarrow{\| \bullet \|} y$$

Thus under the induction hypothesis, are have shown that subspace M is complete with respect to an arbitrary norm which immediately implies that it is closed.

Furthermore, from the above, we see that this statement will be true for any finite dimensional subspace of a normed space. Consider the nth basis vector x_n now and from the set

$$x_n + M = \{x_n + z/z \in M\}$$

Since for any $y, z \in M$,

$$\|x_n + z - (x_n + y)\| = \|z - y\|$$

Since $x_n + M$ is seen to be isometric to M under the mapping $z \in x_n + z$. Hence since M is closed, $x_n + M$ must be closed as well which implies that $C(x_n + M)$ is open, [where $C(x_n + M)$ is the complement of $x_n + M$] we now contend that

$$0 \notin x_n + M$$

for if it did, we would be able to write for some $\beta_1, \beta_2, \ldots, \beta_{n-1} \in F$, $0 = x_n + \beta_1 x_1 + \beta_2 x_2 + \ldots + \beta_{n-1} x_{n-1}$, which is ridiculous. Also 0 is a point of the open set $C(x_n + M)$; Hence there must be a whole nbd of zero lying entirely within $C(x_n + M))$. In other words, there must exist $C_n > 0$ such that for any

$$x \in x_n + M, \|x - 0\| \geq C_n, 0 \in C(x_n + M)$$

[Note that here we say that the distance from any point $x_n + M$ to zero is positive].

Thus for all $\alpha_i \in F (i = 1, \ldots, n-1)$,

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_{n-1} x_{n-1} + x_n\| \geq C_n$$

or

$$\|\frac{\alpha_1}{\alpha_n} x_n + \ldots + \frac{\alpha_{n-1}}{\alpha_n} x_{n-1} + x_n\| \geq C_n$$

which implies for any $\alpha_n \in F$, that

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq \alpha_n C_n$$

because we can write for $\alpha_n \neq 0$.

Suppose now that we had not taken

$$M = < x_1, x_2, \ldots, x_{n-1} >$$

but had taken instead

$$< x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n >$$

since the only fact about M was that its dimension was $n-1$. It is clear that in an analogous fashion we could have arrived at some $c_i > 0$ such that

$$\|\alpha_1 x_1 + \ldots + \alpha_n x_n\| \geq C_i |\alpha_i|$$

for any $i = 1, 2, \ldots, n$. In view of this we can say for any

$$x = \sum_{i=1}^{n} \alpha_i x_i,$$
$$\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n\| \geq \min_i C_i \min_i |\alpha_i| = \min_i C_i \|x\|_0$$

This completes the proof of since $a = \min_i C_i$ is positive.

Corollary: If X is any finite dimensional normed linear space, X is complete [since all norms are equivalent].

Corollary: If X is a normed linear space and M is any finite dimensional subspace, M is closed.

Theorem 6.18. Suppose $A : X \to Y$, where X and Y are normed linear spaces. If X is finite dimensional, A is bounded.

Proof. Suppose dim $X = n$, that a basis for M is given by

$$x_1, x_2, \ldots, x_n.$$

In view of this for any $x \in X$, scalars $\alpha_1, \alpha_2 \ldots, \alpha_n$ such that

$$x = \sum_{i=1}^{n} \alpha_i x_i,$$

and A is linear, we have

$$Ax = \sum_{i=1}^{n} \alpha_i A x_i$$

Letting $K = \sum_{i=1}^{n} \|Ax_i\|$, we have

$$\begin{aligned}\|Ax\| &= \|\sum_{i=1}^{n} \alpha_i A x_i\| \\ &\leq \sum_{i=1}^{n} |\alpha_i| \|Ax_i\|\end{aligned}$$

$$\leq \|x\|_0.K.$$

since $\|x\|_0 = \max_i |\alpha_i|$.

Since all norms in a finite dimensional space are equivalent and A is bounded with respect to zeroth norm, it follows that A must be a bounded linear transformation no matter what norm is chosen for X.

CHAPTER 7

Weak, Strong Convergence and Compactness on Linears Operators

Definition 7.1. If $\|T_n - T\| \to 0$, then we say that the sequence $< T_n >$ of operators (or linear transformation) converges to T and this convergence's is called **convergence in norm** or **strong convergence**. The linear transformation T is said to be the strong limit of the sequence $< T_n >$. Also $< T_n >$ is said to converge weakly towards the linear transformation T if the sequence $< T_n(x) >$ converges to Tx.

Definition 7.2. Let E be a normed linear space, $< T_n >$ a sequence of elements of E and $x_0 \in E$. If the sequence $f(x_n) \to f(x_0)$ as $n \to \infty$ for all functionals $f \in E^*$, then $< T_n >$ is said to converge weakly to x_0 and we write

$$x_n \xrightarrow{w} x_0.$$

x_0 is called the weak limit of the sequence $< T_n >$.

Remark: A sequence can not converge weakly to two different limits, that is the weak limit of a sequence is unique.

We suppose that $x_n \xrightarrow{w} x_0$. and $x_n \to y_0$ i.e $f(x_n) \to f(x_0)$ and $f(x_n) \to f(y_0)$ for an arbitrary linear f. Then

$$f(x_n) = f(y_0)$$

or

$$f(x_0 - y_0) = 0,$$

Now if we choose an f_0 with $\|f_0\| = 1$ and $f_0(x_0 - y_0) = \|x_0 - y_0\|$, then we have $f(x_0 - y_0) = 0$ i.e. $x_0 = y_0$

Proposition: Let N be a normed linear space and $(x_n) \subseteq N$. Then $x_n \to x$ in norm implies $x_n \xrightarrow{w} x$.

Proof.

$$\begin{aligned} |f(x_n) - f(x)| &= |f(x_n - x)| \\ &\leq \|f\| \, \|x_n - x\| \to 0 \,\text{as}\, n \to \infty \\ &[\text{since } x_n \to x \text{ in norm } \forall f \in N^*] \end{aligned}$$

$$\Rightarrow \quad x_n \xrightarrow{w} x.$$

Remark. Thus by above prop, norm convergence or strong convergence $\Rightarrow$ weak convergence.

But the weak convergence need not imply strong convergence. However in a finite dimensional normed linear space, the two convergences are equivalent.

Theorem 7.3. In a finite dimensional space, the notion of weak and strong convergence are equivalent.

Proof: Since strong convergence $\Rightarrow$ weak convergence always.
For the converse suppose $< T_n >$ converges weakly i.e. $f(x_n) \to f(x) \forall f \in E^*$ and E is of finite dimensional. Since E is finite dimensional, $\exists$ a finite system of linearly independent elements $e_1, e_2, \ldots, e_k$ and every $x \in E$ can be represented in the form

$$x = \xi_1 e_1 + \xi_2 e_2 + \ldots + \xi_k e_k$$

with real $\xi_1, \xi_2, \ldots, \xi_k$. Thus

$$x_n = \xi_1^{(n)} e_1 + \xi_2^{(2)} e_2 + \ldots + \xi_k^{(n)} e_k$$

Now we consider such functionals $f_i \in E^*$ for which $f_i(e_i) = 1$ and $f_i(e_k) = 0$ for $k \neq i$. Thus

$$f_i(x_n) = \xi_i^{(n)} \text{ and } f_i(x_0) = \xi_i^{(0)}$$

But since the sequence $f(x_n) \to f(x_0)$ for every linear functional f, so also $f_i(x_n) \to f_i(x_0)$ that is

$$\xi_i^{(n)} \to I_{u_i}^0 \text{ for } i = 1, 2, \ldots, k$$

Let M be the greatest of the numbers $\|e_i\|, (i = 1, 2, \ldots, k)$ i.e. $M = \max \|e_i\|$.

Then for any given $\in > 0$, $\exists$ an n_0 such that

$$|\xi_i^{(n)} - \xi_i^{(0)}| < \frac{\in}{M.K}$$

for all $i = 1, 2, \ldots, k$ and $n \geq n_0$. Thus

$$\begin{aligned}
\|x_n - x_0\| &= \| \sum_{i=1}^{n} (\xi_i^{(n)} - \xi_i^{(0)}) e_i \| \\
&\leq \sum_{i=1}^{n} |(\xi_i^{(n)} - \xi_i^{(0)})| \|e_i\|
\end{aligned}$$

$$<\in.$$

Hence the sequence $< x_n >$ converges strongly to x_0.

Compact Operation on Normed Spaces

Definition 7.4. Let X and Y be normed spaces. An operator $T : X \to Y$ is called a compact linear operator (or completely continuous linear operator) if T is linear and if for every bounded subset M of X, the image $T(M)$ is relatively compact that is the closure $\overline{T(M)}$ is compact.

Remark: Many linear operators in analysis are compact. A systematic theory of compact linear operators emerged from the theory of integral equations of the form

$$(T-\lambda I)x(s) = y(s) \quad \text{where } Tx(s) = \int_a^b K(s,t)\,x(t)\,dt.$$

where λ is a parameter, Y and kernel K are given functions (subject to certain conditions) and x is the unknown function. Such equations also play a role in the theory of ordinary and partial differential equations. The term compact is suggested by the definition. The older term completely continuous can be motivated by the following Lemma which shows that a compact linear operator is continuous but the converse is not generally true.

Relation of Compact and Continuous Linear Operator

Theorem 7.5. Let X and Y be normed spaces. Then

(a) Every compact linear operator $T : X \to Y$ is bounded, hence continuous

(b) If $\dim X = \infty$, the identity operator $I : X \to Y$ (which is continuous) is not compact.

Proof: (a) Since the unit sphere $U = \{x \in X : \|x\| = 1\}$ is bounded and T is compact, so by definition $\overline{T(U)}$ is compact. Now since every normed space is metric space and by the result "Every compact subset of a metric space is closed and bounded." so that

$$\sup_{\|x\|=1} \|Tx\| < \infty.$$

Hence T is bounded. But by the result "Let $T : D(T) \to Y$ be a linear operator, where $D(T) \subset X$ and X, Y are normed spaces. Then

(1) T is continuous if and only if T is bounded.

(2) If T is continuous at a single point, T is continuous".

Thus T is continuous. Hence every compact linear operator $T : X \to Y$ is bounded and hence continuous.

(b) Since the closed unit ball $M = \{x \in X; \|x\| \leq 1\}$ is bounded. If $\dim X = \infty$, then by the result "If a normed space X has the property that the closed unit ball $M = \{x; \|x\| \leq 1\}$ is compact, then X is finite dimensional" M can not be compact. Thus $I(M) = M = \overline{M}$ is not relatively compact.

Remark. From the definition the compactness of a set, we obtain a useful criterion for operators.

Theorem 7.6. Let X and Y be normed spaces and $T : X \to Y$ be linear operator. Then T is compact if and only if it maps every bounded sequence $< x_n >$ in X onto a sequence $< x_n >$ in Y which has a convergent subsequence.

Proof: If T is compact and $< x_n >$ is bounded, then the closure of $< x_n >$ in X is compact. Since every normed space is metric space and by the definition, "a metric space X is said to be compact if every sequence in X has a convergent subsequence". Thus $< x_n >$ contains a convergent subsequence.

Conversely assume that every bounded sequence $< x_n >$ contains a subsequence $< x_{nk} >$ such that $< Tx_{nk} >$ converges in Y. Consider any bounded subset $B \subset X$, and let $< y_n >$ be any sequence in $T(B)$. Then $y_n = Tx_n$ for some $x_n \in B$ and $< x_n >$ is bounded since B is bounded. But by assumption $< Tx_n >$ contains a convergent subsequence. Hence by definition of compactness, $\overline{T(B)}$ is compact. Since $< y_n >$ in $T(B)$ was arbitrary. Thus by definition of compact operator, T is compact.

Remark: The sum $T_1 + T_2$ of two compact linear operators from normed space X to normed space X is compact. Similarly αT_1 is compact, where α is any scalar. Thus the compact linear operators from X into X form a vector space.

Theorem 7.7. Let X and Y be normed spaces and $T : X \to Y$ a linear operator. Then

(a) If T is bounded and $\dim T(X) < \infty$, the operator T is compact.

(b) If $\dim X = \infty$, the operator T is compact.

Proof: (a) Let $< x_n >$ be any bounded sequence in X. Then the inequality $\|Tx_n\| \leq \|T\| . \|x_n\|$ shows that $< Tx_n >$ is bounded. Now by the result "In a finite dimensional normed space X, any subset $M \subset X$ is compact if and only if M is closed and bounded" and $\dim(X) < \infty$ implies that $< Tx_n >$ is relatively compact. It follows that $< Tx_n >$ has a convergent subsequence. But by Theorem 2, $T : X \to Y$ is compact if and only if T maps every bounded sequence $< x_n >$ in X onto a sequence $< Tx_n >$ in Y which has a convergent subsequence". Hence the operator T is compact.

(b) Since we know that if a normed space X is finite dimensional then every linear operator on X is bounded operator. Thus T is bounded. Also $\dim X = \infty$. Now by the result "If T is a linear operator and dim $D(T) < \infty$, then $\dim R(T) \leq n$ "where $D(T)$ and $R(T)$ are domain and range of T." Thus if $\dim T(X) = \infty$, then $\dim(X) < \infty$. Now since $\dim T(X) < \infty$ and T is bounded. It follows by (a) part that the operator T is compact.

Compactness of Limit of the Sequence of Compact Operators

Theorem 7.8. Let $< T_n >$ be a sequence of compact linear operators from a normed space X into a Banach space Y. If $< T_n >$ is uniformly operator convergent, say $\|T_n - T\| \to 0$, then the limit operator T is compact.

Proof: Using a diagonal method, we show that for any bounded sequence $< x_m >$ in X, the image $< Tx_m >$ has a convergent subsequence and then apply Theorem 2 i.e. "Let X and Y be normed spaces and $T : X \to Y$, a linear operator. Then T is compact if and only if it maps every bounded sequence $< x_n >$ in X onto a sequence $< Tx_m >$ in Y which has a convergent subsequence."

Since T_1 is compact, $< x_m >$ has a subsequence $< x_{1,m} >$ such that $< T_1 x_{1,m} >$ is Cauchy. Similarly $< x_{1,m} >$ has a subsequence $< x_{2,m} >$ such that $< T_2 x_{2,m} >$ is Cauchy. Continuing in this way, we see that the diagonal sequence $< y_m > = < x_{m,m} >$ is a subsequence of $< x_m >$ such that for every fixed positive integer n, the sequence $< T_n y_m >_{m \in N}$ is Cauchy. $< x_m >$ is bounded, say $\|x_m\| \leq c$ for all m. Hence $\|y_m\| \leq c$ for all m. Let $\in > 0$. Since $T_m \to T$, there is an $n = p$ such that

$$\left\|T - T_p\right\| < \in /3c \tag{1}$$

Since $< T_n y_m >_{m \in N}$ is Cauchy, there is an N such that

$$\left\|T_p y_j - T_p y_k\right\| < \frac{\in}{3} \, (j, k > N) \tag{2}$$

Hence we obtain for $j,k > N$.

$$\begin{aligned}
\|Ty_j - Ty_k\| &\leq \|Ty_j - T_py_j\| + \|T_py_j - T_py_k\| + \|T_py_j - Ty_k\| \\
&\leq \|T - T_p\| \cdot \|y_j\| + \frac{\in}{3} + \|T_p - T\| \cdot \|y_j\| \\
&< \frac{\in}{3c}.c + \frac{\in}{3} + \frac{\in}{3c}.c \quad \text{(Using (1) and (2))} \\
&= \in
\end{aligned}$$

This shows that $< Ty_m >$ is cauchy and converges since Y is complete. But $< y_n >$ is a subsequence of the arbitrary bounded sequence $< x_m >$. Hence using Theorem 2, which states that "Let X and Y be normed spaces and $T : X \to Y$, a linear operator. Then T is compact if and only if it maps every bounded sequence $< x_n >$ in X onto a sequence $< Tx_m >$ in Y which has a convergent subsequence," we get that the operator T is compact.

Remark. The above theorem states conditions under which to limit of a sequence of compact linear operators is compact. This theorem is also important as a tool for proving compactness of a given operator by exhibiting it as the uniform operator limit of a sequence of compact linear operators.

Note that the present theorem becomes false if we replace uniform operator convergence by strong operator convergence $\|T_nx - Tx\| \to 0$. This can be seen from $T_n : l^2 \to l^2$ defined by

$$T_n(x) = (\xi_1, \ldots, \xi_n, 0, 0, \ldots)$$

Where $x = (\xi_i) \in l^2$. Since T_n is linear and bounded, T_n is compact by Theorem 3(a). Clearly $T_nx \to x = Ix$ but I is not compact since $\dim l^2 = \infty$.

The following example illustrates how the theorem can be used to prove compactness of an operator.

Example (space l^2). To prove compactness of $T : l^2 \to l^2$ defined by

$$y = (\eta_j) = Tx$$

where $\eta_j = \xi_j / j$ for $j = 1, 2, \ldots$

Solution. T is linear. If $x = (\xi_j) \in l^2$, then. Let $T_n : l^2 \to l^2$ be defined by

$$T_nx = \left(\xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \ldots, \frac{\xi_n}{n}, 0, 0, \ldots\right).$$

T_n is linear and bounded and is compact by Theorem 3(a), Further

$$\begin{aligned}\|(T-T_n)x\|^2 &= \sum_{j-n+1}^{\infty} |\eta_j|^2 = \sum_{j-n+1}^{\infty} \frac{1}{J^2} \cdot |\xi_j|^2 \\ &\leq \frac{1}{(n+1)^2} \sum_{j=n+1}^{\infty} |\xi_j|^2 \leq \frac{\|x\|^2}{(n+1)^2}\end{aligned}$$

Taking the supremum over all x of norm 1, we see that

$$\|T-T_n\| \leq \frac{1}{n+1}.$$

Hence $T_n \to T$ and hence T is compact by the above Theorem 4.

Theorem 7.9. Let X and Y be normed spaces and $T : X \to Y$ a compact linear operator. Suppose that $< x_n >$ in X is weakly convergent, say, $x_n \xrightarrow{w} x$. Then $< Tx_n >$ is strongly convergent in Y and has the limit $y = Tx$.

Proof. We write $y_n = Tx_n$ and $y = Tx$. First we show that

$$y_n \xrightarrow{w} y. \tag{1}$$

Then we show that

$$y_n \to y \tag{2}$$

Let g be any bounded linear functional on Y. We define a functional f on X by setting

$$f(z) = g(Tz) \quad (z \in X)$$

f is linear and bounded because T is compact, hence

$$|f(z)| = |g(Tz)| \leq \|g\| \cdot \|Tz\| \leq \|T\| \cdot \|z\|$$

By definition $x_n \xrightarrow{w} x$ implies $f(x_n) \to f(x)$, hence by the definition, $g(Tx_n) \to g(Tx)$, that is, $g(y_n) \to g(y)$ since g was arbitrary, this implies that $y_n \xrightarrow{w} y$. which proves (1).

Now we prove (2). Assume that (2) does not hold. Then $< y_n >$ has a subsequence $< y_{nk} >$ such that

$$\|y_{nk} - y\| \geq \eta \tag{3}$$

for some $\eta > 0$. Since $< x_n >$ is weakly convergent, by the result "Let $< x_n >$ be a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$, then the sequence $< ||x_n|| >$ is bounded". Thus $< x_n >$ is bounded and so is $< y_{nk} >$. But by Theorem 2, "Let X and Y be normed spaces and $T : X \to Y$, a linear operator.
Then T is compact if and only if it maps every bounded sequence $< x_n >$ in X onto a sequence $< Tx_n >$ in Y which has a convergent subsequence", since the operator T is compact, $< Tx_{nk} >$ has a convergent subsequence say $< \bar{y}_j >$. Let $\bar{y}_j \to \bar{y}$. Hence $\bar{y}_j \xrightarrow{w} \bar{y}$. Since by the result " Let $< x_n >$ be a weakly convergent sequence in a normed space X, say $x_n \xrightarrow{w} x$, then every subsequence of $< x_n >$ converges weakly to x ", Thus by this result and (1) we have $\bar{y} = y$. consequently

$$||\bar{y} - \bar{y}|| \to 0$$

But

$$||\bar{y}_i - y|| \geq \eta > 0 \qquad [by(3)]$$

This contradicts, so that (2) must hold.

Closed Range Theorem

Definition 7.10. Suppose X is a Banach space, M is a subspace of X and N is a subspace of X^* (Dual space of X), neither M nor N is assumed to be closed.

Their annihilators $M^{\perp}$ and $N^{\perp}$ are defined as follows:

$$M^{\perp} = \{x^* \in X^*, < x, x^* >= 0 \text{ for all } x \in M\}$$

$$N^{\perp} = \{x \in X, < x, x^* >= 0 \text{ for all } x^* \in M\}$$

Thus $M^{\perp}$ consists of all bounded linear functionals on X that vanish on M and $N^{\perp}$ is the subset of X on which every member of N vanishes. It is clear that $M^{\perp}$ and $N^{\perp}$ are vector spaces. Since M is the intersection of the null spaces of the functionals, $M^{\perp}$ is a weak* closed subspace of X^*.

The weak*-topology of X^* is by definition, the weakest one that makes all functionals

$$x^* \to < x, x^* >$$

continuous. Thus the norm topology of X^* is stronger than its weak*-topology.

Notation. If T maps X into Y, then the null space of T and range of T will be denoted by $N(T)$ and $\Re(T)$ respectively

$$N(T) = \{x \in X, Tx = 0\}$$

$$\Re(T) = \{y \in Y; Tx = y \text{ for some } x \in X\}$$

Theorem 7.11. If X and Y are Banach spaces and if $T \in B(X,Y)$ [set of bounded or continuous linear operator], then each of the following three conditions implies the other two:

(a) $\Re(T)$ is closed in Y.

(b) $\Re(T^*)$ is weak*-closed in X^*.

(c) $\Re(T^*)$ is norm-closed in X^*.

Proof: It is obvious that (b) implies (c). We will prove that (a) implies (b) and that (c) implies (a).

Suppose (a) holds. Then $N(T)^{\perp}$ is the weak closure of $R(T^*)$.

To prove (b), it is therefore enough to show that

$$N(T)^{\perp} \subset \Re(T^*)$$

Pick $x^* \in N(T)^{\perp}$. Define a linear functional Λ on $\Re(T)$ by

$$\Lambda Tx = < x, x^* > \quad (x \in X)$$

Note that is well defined for if $Tx = Tx'$, then $x - x' \in N(T)$, hence

$$< x - x', x^* > = 0$$

The open mapping theorem applies to

$$T : X \to \Re(T)$$

since $\Re(T)$ is assumed to be a closed subspace of the complete space Y and is therefore complete. It follows that there exists $K < \infty$ such that to each $y \in \Re(T)$ corresponds an $x \in X$ with $Tx = y$, $\|x\| \leq K\|y\|$ and

$$|\Lambda y| = |\Lambda Ty| = |< x, x^* >| \leq K\|y\| . \|x^*\|$$

Thus is Λ continuous. By the Hahn-Banach theorem some $y^* \in Y^*$ extends Λ. Hence

$$< Tx, y^* > = \Lambda Tx = < x, x^* > (x \in X)$$

This implies $x^* = T^*y^*$. Since X^* was an arbitrary element of $N(T)^{\perp}$, we have shown that

$$N(T)^{\perp} \subset \Re(T^*)$$

Thus (b) follows from (a).

Suppose next that (c) holds. Let Z be the closure of $\Re(T)$ in Y. Define some $S \in B(X,Z)$ by setting $Sx = Tx$. Since (S) is dense in Z. Thus $S^* : Z^* \to X^*$ is one-to-one. If $z^* \in Z^*$, then by Hahn-extensions theorem, we get an extension y^* of z^*, for every $x \in X$,

$$< x, T^*y^* >=< Tx, y^* >< Sx, y^* >=< x, S^*z^* >$$

Hence $S^*z^* = T^*y^*$. It follows that S^* and T^* have identical ranges. Since (c) is assumed to hold. $\Re(S^*)$ is closed, hence complete. Apply the open mapping theorem to

$$S^* : Z^* \to \Re(S^*)$$

Since S^* is one to one, the conclusion is that there is a constant $c > 0$ which satisfies

$$c\|z^*\| \leq \|S^*z^*\|$$

for every $z^* \in Z^*$.

Now using the following result

"Suppose U and V are the open unit balls in the Banach space X and Y, respectively. Suppose $T \in B(X,Y)$ and $C > 0$,

(a) If the closure of $T(U)$ contains cV, then

$$T(U) \supset cV$$

(b) If $c\|y^*\| \leq \|T^*y^*\|$ for every $y^* \in Y^*$, then

$$T(U) \supset cV.$$

CHAPTER 8

Inner Product and Hilbert Spaces

The notion of dot product and the condition of orthoganality are totally missing in normed linear space, extention of these notions to any arbitrary infinite dimensional linear space, leads to definition of inner product space on linear space in such a way that inner product give rise to norm. The inner product spaces are special normed linear spaces. A complete normed linear spaces is called Hilbert space. Also every Hilbert space is a Banach space but not conversely.

The study of Hilbert space includes Schwarz, parallelogram and polrization inequalities. Further defining orthoganal complements and establishing orthoganal decomposition theorem guarantees that there are are plenty of projections in a Hilbert space. The chapter conludes with proof of projection theorem and Bessel's inequality.

Definition 8.1. An inner product space X or pre-Hilbert space is a complex linear space together with an inner product $(.,.) : X \otimes X \to C$ such that

(i) $(x,y) = \overline{(y,x)}$, $\forall x,y \in X$

(ii) $(\lambda x + \mu y, z) = \lambda\,(x,z) + \mu\,(y,z)$, $\forall x,y,z \in X$ and $\lambda, \mu \in X$

(iii) $(x,x) \geq 0$ and $(x,x) = 0$ iff $x = 0$, $\forall x \in X$

condition (i) clearly reduces to $(x,y) = (y,x)$ if X is real vector space. From (i) and (ii), we obtain

$$\begin{aligned}(x, cy+dz) &= \left(\overline{cy+dz,x}\right)\\ &= \overline{c\,(y,x)} + \overline{d\,(z,x)}\\ &= \bar{c}\,(y,x) + \bar{d}\,(z,x)\end{aligned}$$

In any pre-Hilbert space, the following are immediate

(a) $(x, y+z) = (x,y) + (x,z)$

(b) $(x, \lambda y) = \bar{\lambda}\,(x,y)$

(c) $(0,y) = (x,0) = 0$

(d) $(x-y, z) = (x,z) - (y,z)$

Examples

1. Let C^n be the vector space of n tuples. If $x = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $y = (\mu_1, \ldots, \mu_n)$ define

$$(x,y) = \sum_{k=1}^{n} \lambda_k \overline{\mu_k}$$

Then all the axioms for pre-Hilbert space are satisfied. This example is known as n-dimensional unitary space and will be denoted by C^n. In this space, the norm of x is defined by

$$\|x\| = \left(\sum_{i=1}^{n} |\lambda_i|^2\right)^{1/2}$$

2. Let $C(a,b)$ be the vector space of continuous functions defined on $[a,b], a < b$. Define

$$(x,y) = \int_a^b x(t).\overline{y(t)}dt$$

With respect to this inner product, $C[a,b]$ is a pre-Hilbert space. The norm of x in $C[a,b]$ is introduced by taking

$$\|x\| = \left(\int_a^b |x(t)|^2 dt\right)^{1/2}$$

3. Let P be the vector space of finitely non-zero sequences. If $x = (\lambda_k)$ and $y = (\mu_k)$, define

$$(x,y) = \sum_{k=1}^{n} \lambda_k \overline{\mu_k}$$

This space is a pre-Hilbert space with respect to this inner product. The norm of x in this space is defined by

$$\|x\| = \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{1/2}$$

Theorem 8.2. Each Inner Product space is a normed linear space under $\|x\| = (x,x)^{1/2}$.

Since all the properties of norm are satisfied. We notice that

(a) $\|x\| = (x,x)^{1/2} \geq 0$

(b) $\|x\| = 0 \Leftrightarrow (x,x) = 0$ iff $x = 0$

(c) $\|\alpha x\|^2 = (\alpha x, \alpha x)$

$$= \alpha\overline{\alpha}\,(x,x)$$

$$= |\alpha|^2\,\|x\|^2$$

$$\Rightarrow \|\alpha x\| = |\alpha|\;\|x\|$$

(d) For $x, y \in X$, we have

$$\|x+y\|^2 = (x+y, x+y) \equiv (x, x+y) + (y, x+y)$$

$$= \|x,x\| + (y,x) + (x,y) + (y,y)$$

$$= \|x,x\| + (y,y) + (x,y) + (\overline{x,y})$$

$$= \|x,x\| + (y,y) + 2R\,(x,y)$$

$$\leq \|x\|^2 + \|y\|^2 + 2\,\|x\|\;\|y\|$$

$$= (\|x\| + \|y\|)^2$$

$$\Rightarrow \|x+y\| \leq \|x\| + \|y\|$$

Therefore, each pre-Hilbert space is a normed linear space.

Theorem 8.3. The Inner product (Scalar Product) is a continuous function with respect to norm convergence. (**Inner Product in an Hilbert space is jointly continuous**).

Proof. If $x_n \to x$ and $y_n \to y$, then the number $\|x_n\|, \|y_n\|$ are bounded. Let M be their upper bound. Then

$$\begin{aligned}
|(x_n, y_n) - (x,y)| &= |(x_n, y_n) - (x_n, y) + (x_n, y) - (x,y)| \\
&\leq |(x_n, y_n) - (x_n, y)| + |(x_n, y) - (x,y)| \\
&= |(x_n, y_n - y) + (x_n - x, y)| \\
&\leq \|x_n\|\;\|y_n - y\| + \|x_n - x\|\;\|y\| \quad \text{(by Schwarz inequality)} \\
&\leq M\;\|y_n - y\| + \|y\|\;\|x_n - x\|
\end{aligned}$$

Now since $\|y_n - y\| \to 0$ and $\|x_n - x\| \to 0$ as $n \to \infty$, therefore $|(x_n - y_n) - (x,y)| \to 0$ for $n \to \infty$ and hence $(x_n - y_n) \to (x,y)$. Thus inner product in a pre-Hilbert space is jointly continuous.

Theorem 8.4. (Cauchy-Schwarz Inequality). If x and y are any two vectors in an inner product space, then

$$|(x,y)| \leq \|x\| \ \|y\|.$$

Proof. We have $(x+\lambda y, x+\lambda y) \geq 0$ for arbitrary complex λ.

$$\Rightarrow \quad (x, x+\lambda y) + \lambda\,(y, x+\lambda y) \geq 0$$

$$\Rightarrow \quad (x,x) + \overline{\lambda}\,(x,y) + \lambda\left[(y,x) + \overline{\lambda}\,(y,y)\right] \geq 0.$$

$$\Rightarrow \quad (x,x) + \overline{\lambda}\,(x,y) + \lambda\left[(y,x) + \lambda\overline{\lambda}\,(y,y)\right] \geq 0$$

if we put is $\lambda = \frac{-(x,y)}{(y,y)}$, then

$$(x,x) - \frac{\overline{(x,y)}\,(x,y)}{(y,y)} - \frac{(x,y)\,(y,x)}{y,y} + \frac{(x,y)\,\overline{(x,y)}\,(y,y)}{(y,y)} \geq 0$$

$$\Rightarrow \quad (x,x) - \frac{|(x,y)|^2}{(y,y)}\frac{(x,y)\,(y,x)}{y,y} + \frac{(x,y)\,(y,x)}{(y,y)} \geq 0$$

$$\Rightarrow \quad (x,x) - \frac{|(x,y)|^2}{(y,y)} \geq 0$$

$$\Rightarrow \quad |(x,y)|^2 \leq (x,x)\,(y,y) = \|x\|^2 . \|y\|^2$$

$$\Rightarrow \quad |(x,y)| \leq \|x\| \ \|y\|$$

Theorem 8.5. (Parallelogram Law). In an Hilbert space H,

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2 \quad \forall\, x,y \in H.$$

Proof. Writing out the expression on the left in terms of inner products.

$$\begin{aligned}
\|x+y\|^2 + \|x-y\|^2 &= (x+y, x+y) + (x+y, x+y) \\
&= (x,x) + (x,y) + (y,x) + (y,y) + (x,x) - (x,y) - (y,x) + (y,y) \\
&= 2\,(x,x) + 2\,(y,y) \\
&= 2\,\|x\|^2 + 2\,\|y\|^2
\end{aligned}$$

Polarization Identity

Theorem 8.6. In a pre-Hilbert space, (inner-product space)

$$4\,(x,y) = \left[\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2\right]$$

Proof. We note that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + (x,y) + (y,x) \tag{1}$$

Replace y by $-y, iy$ by $-iy$ and obtain

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - (x,y) - (y,x) \tag{2}$$

and

$$\|x+iy\|^2 = \|x\|^2 + \|y\|^2 - i\,(x,y) + i\,(y,x) \tag{3}$$

$$\|x+iy\|^2 = \|x\|^2 + \|y\|^2 - i\,(x,y) - i\,(y,x) \tag{4}$$

It follows that

(2) $-\|x-y\|^2 = -\|x\|^2 - \|y\|^2 + (x,y) + (y,x)$
(3) $i\,\|x+iy\|^2 = i\,\|x\|^2 + i\,\|y\|^2 + (x,y) - (y,x)$
(4) $-i\,\|x-iy\|^2 = -i\,\|x\|^2 - i\,\|y\|^2 + (x,y) - (y,x)$

Adding (1), (2), (3) and (4), we get

$$\|x+y\|^2 - \|x-y\|^2 + i\,\|x+iy\|^2 - i\,\|x-iy\|^2 = 4\,(x,y)$$

This completes the proof.

Definition 8.7. A complete pre-Hilbert space (Inner Product space) is called Hilbert space. Thus a Banach space whose norm is generated by inner product is called Hilbert space.

Example 8.8. Denote by H, the set of all sequences $x = (\lambda_k)$ of complex number such that

$$\sum_{k=1}^{\infty} |\lambda_i|^2 < \infty$$

If $x = (\lambda_k)$ and $y = (\mu_k)$ are sequences belonging to H, then by the parallelogram law for complex numbers,

$$|\lambda_k + \mu_k|^2 + |\lambda_k + \mu_k|^2 = 2\,|\lambda_k|^2 + 2\,|\mu_k|^2$$

Hence

$$\sum_{k=1}^{n} |\lambda_k + \mu_k|^3 \leq 2\sum_{k=1}^{n} |\lambda_k|^2 + 2\sum_{k=1}^{n} |\mu_k|^2$$

for all n. Hence $\sum_{k=1}^{n} |\lambda_k + \mu_k|^2 < \infty$ by the comparison test. Hence the sequence $(\lambda_k + \mu_k)$ belongs to H, that is $x + y \in H$. Furthermore if $x = (\lambda_k)$ belongs to and H is a complex number, then

$$\sum_{k=1}^{n} |\lambda\lambda_k|^2 = |\lambda|^2 \sum_{k=1}^{n} |\lambda_k|^2$$

shows that the sequence $(\lambda\lambda_k)$ is absolutely summable, it is denoted by λx.

With respect to the operations $x + y$ and λx, H becomes a linear space. We also note that if $x = (\lambda_k)$ and $y = (\mu_k)$ belong to H, then the series

$$\sum_{k=1}^{n} \lambda_k \overline{\mu_k}$$

converges absolutely. In fact, a and b are real numbers, $(a - b)^2 \geq 0$ leads to $ab \leq \frac{1}{2}\left(a^2 + b^2\right)$ and in particular, we have

$$|\lambda_k \overline{\mu_k}| \leq \frac{1}{2}\left(|\lambda_k|^2 + |\overline{\mu_k}|^2\right.$$

Thus $\sum_{k=1}^{n} \lambda_k \overline{\mu_k}$ converges by the comparison test.

This justifies the definition of the inner product for H as

$$(x, y) = \sum_{k=1}^{\infty} \lambda_k \overline{\mu_k}$$

The axioms for a pre-Hilbert space are easily verified. The norm of an element x in this space is defined by

$$\|x\| = \left(\sum_{k=1}^{\infty} |\lambda_k|^2\right)^{\frac{1}{2}}$$

It can be seen that

$$\|\lambda x\| = |\lambda| \,.\, \|x\|$$

and that

$$\|x+y\|^2 + \|x-y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$$

Thus to prove that H is a Hilbert space, it is sufficient to show that H is complete.

Suppose $x_1, x_2, \ldots,$ is a Cauchy sequence in H, that is $\|x_m - x_n\| \to 0$ as $m,n \to \infty$, say $x_n = \left(\lambda_k^n\right)$ for each k, $\left|\lambda_k^m - \lambda_k^n\right| \le \sum_{j=1}^{\infty} \left|\lambda_J^m - \lambda_J^n\right|^2 = \|x_m - x_n\|^2$ shows that the sequence $\lambda_k^1, \lambda_k^2, \ldots,$ of k th components is Cauchy. Since the set of complex numbers is complete, $\lambda_k^n \to \lambda_k$ as $n \to \infty$ for suitable λ_k. It will be shown that $\sum_{k=1}^{\infty} |\lambda_k|^2 < \infty$ and that $< x_n >$ converges to $x = (\lambda_k)$.

Let $\in > 0$ be given. Let p be an index such that $\|x_m - x_n\|^2 \le \in$ whenever $m,n \ge p.$.

Fix any positive integer r, then we have

$$\sum_{k=1}^{\infty} |\lambda_k^m - \lambda_k^n|^2 \le \|x_m - x_n\|^2 \le \in$$

Provided $m,n \ge p$. Letting $m \to \infty$,

$$\sum_{k=1}^{\infty} |\lambda_k - \lambda_k^n|^2 \le \in$$

provided $n \ge p$, since r is arbitrary, we get

$$\sum_{k=1}^{\infty} |\lambda_k - \lambda_k^n|^2 \le \in \quad \text{whenever } n \ge p \tag{1}$$

In particular, $\sum_{k=1}^{\infty} \left|\lambda_k - \lambda_k^n\right|^2 \le \in.$

Hence the sequence $< \lambda_k - \lambda_k^p >$ belongs to H. Adding to it, the sequence $< \lambda_k^p >$ of H, we obtain $(\lambda_k) = x$ belongs to H. It follows from (1) that $\|x - x_n\|^2 \le \in$ whenever $n \ge p$. Thus $x_n \to x$ and hence H is complete. This Hilbert space of absolutely square summable sequences is denoted by l^2.

Theorem 8.9. In a pre-Hilbert space, every cauchy sequence is bounded.

Proof. Let $< x_n >$ be a cauchy sequence and let N be an index such that $\|x_n - x_m\| \le 1$ whenever $m,n \ge N$. If $n \ge N$, then

$$\begin{aligned} \|x_n\| &= \|(x - x_N) + x_N\| \\ &\le \|x - x_N\| + \|x_N\| \end{aligned}$$

$$\leq 1 + \|x_N\|$$

Thus if M is the largest of the numbers $1 + \|x_N\|, \|x_1\|, \ldots, \|x_{N-1}\|$,, we have $\|x_n\| \leq M$ for all n. Hence $< x_n >$ is bounded.

Theorem 8.10. In any pre-Hilbert space, if $< x_n >$ and $< y_n >$ are Cauchy sequence of vectors, then $< (x_n, y_n) >$ is Cauchy (hence convergent) sequence of scalars.

Proof. By Cauchy-Schwarz inequality

$$\begin{aligned}
&|(x_n, y_n) - (x_m, y_m)| \\
&= |(x_n - x_m, y_n - y_m) + (x_m, y_n - y_m) + (x_n - x_m, y_m)| \\
&\leq |(x_n - x_m, y_n - y_m) + (x_m, y_n - y_m) + (x_n - x_m, y_m)| \\
&\leq \|(x_n - x_m\| . \|y_n - y_m\| + \|x_m\| . \|y_n - y_m\| + \|x_n - x_m\| . \|y_m\|
\end{aligned}$$

for all m and n. Since $\|x_m\|$ and $\|y_m\|$ are bounded. Therefore by the above theorem, R.H.S. of the above inequality $\to 0$ and $m, n \to \infty$. Therefore $< (x_n, y_n) >$ is cauchy sequence of scalars and hence convergent.

Remark 8.11. It follows from this theorem, that in a pre-Hilbert space if $< x_n >$ is a Cauchy sequence, then (x_n, x_n) and hence $\|x_n\|$ is a cauchy sequence of scalars, and hence convergent.

It is clear from the definition that every Hilbert space is a Banach space. We shall see that converse need not be true. The question arises under what condition, a Banach space will become a Hilbert space. In this direction, we have the following result.

Theorem 8.12. A Banach space is a Hilbert space $\Leftrightarrow \|gm$ (parallelogram) law holds.

Proof. Let H be a Hilbert space. Thus it is by definition, a Banach space whose norm arises from the inner product taken as $\|x\| = (x, x)^{1/2}$

Then

$$\begin{aligned}
&\|x + y\|^2 + \|x - y\|^2 \\
&= (x + y, x + y) + (x - y, x - y) \\
&= (x.x) + (y, y) + (x, y) + (y, x) + (x, x) + (y, y) - (x, y) - (y, x) \\
&= 2(x, x) + 2(y, y) \\
&= 2\|x\|^2 + \|y\|^2 .
\end{aligned}$$

Thus if H is a Hilbert space, then it is a Banach space satisfying $\|gm$ law.

Conversely: Suppose that H is a Banach space and that in $H, \|gm$ law holds good.

We define an inner product in H by

$$(x,y) = \frac{1}{4}\left[\|x+y\|^2 - \|x+y\|^2\right] \tag{1}$$

Then $(x,x) \geq 0$ and $(x,x) = 0 \Leftrightarrow x = 0$ Moreover $(x,x) = \|x\|^2$ and $(x,y) = (y,x)$.

It is only to show that

$$(x_1 + x_2, y) = (x_1, y) + (x_2, y)$$

and

$$(\alpha x, y) = \alpha (x,y)$$

by $\|gm$ law, we note that

$$\|u+v+w\|^2 + \|u+v-w\|^2 = 2\|u+v\|^2 + 2\|w\|^2$$

and

$$\|u-v+w\|^2 + \|u-v-w\|^2 = 2\|u-v\|^2 + 2\|w\|^2.$$

so that on substracting

$$\begin{aligned}
&\|u+v+w\|^2 + \|u+v-w\|^2 - \|u-v+w\|^2 - \|u-v-w\|^2 \\
&\qquad = 2\|u+v\|^2 - 2\|u-v\|^2 \\
&\Rightarrow (u+w,v) + (u-w,v) = 2(u,v) \quad \text{[using(1)]} \\
&\qquad = (2u,v) \qquad (2)
\end{aligned}$$

Setting $u = w$, this implies $(2u,v) = (u,v)$. Now let $x_1 = u+w, x_2 = u+w$ and $y = v$ to obtain

$$(x_1,y) + (x_2,y) = (x_1+x_2,y) \quad \text{[using 2]}$$

Similarly, $(ax,y) = a(x,y)$. Thus a Banach space satisfying $\|gm$ is a Hilbert space.

Example of a Banach space which is not Hilbert space

Example 8.13. We know that a Banach space is a Hilbert space if and only if $\|gm$ Law holds.

Consider the linear space $L_1(0,1)$ consisting of equivalence classes of functions summable on $[0,1]$ w.r. to Lebesgue measure with the norm of $f \in L_1[0,1]$ as

$$\|f\| = \int_0^1 |f(x)|\,dx \tag{1}$$

$L_1[0,1]$ is a Banach space under this norm.

We show that this norm does not satisfy $\|$ law and thus precludes the possibility of viewing this space as a Hilbert space.

Consider the sets $A = \left[0, \frac{1}{2}\right]$ and $B = \left[\frac{1}{2}, 1\right]$ and the characteristic functions of these sets χ_A and χ_B. We note that (1) yields.

$$\begin{aligned}
\|\chi_A + \chi_B\|^2 &= \left(\int_0^1 |\chi_A + \chi_B|\right)^2 \\
&= \left(\int_0^{1/2} |\chi_A + \chi_B| + \int_{1/2}^0 |\chi_A + \chi_B|\right)^2 \\
&= \left[\frac{1}{2} + \frac{1}{2}\right]^2 = l^2 = 1 \\
\|\chi_A + \chi_B\| &= \left(\left(\int_0^1 |\chi_A + \chi_B|\right)^2\right)^{1/2} \\
&= \left(\int_0^{1/2} |\chi_A + \chi_B| + \int_{1/2}^0 |\chi_A + \chi_B|\right)^2 \\
&= \left[\frac{1}{2} + \frac{1}{2}\right]^2 = 1
\end{aligned}$$

But

$$2\|\chi_A\|^2 + \|\chi_B\|^2 = 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Thus

$$\|\chi_A + \chi_B\|^2 + \|\chi_A - \chi_B\|^2 \neq 2\|\chi_A\|^2 + 2\|\chi_B\|^2$$

and therefore $\|gm$ Law is not satisfied and hence $L_1[0,1]$ is not a Hilbert space.

Definition 8.14. A convex set in a Banach space. B is a non empty subset S such that $x, y \in S \Rightarrow x(1-t) + ty \in S$ for every real number t satisfying $0 < t < 1$.

If we put $t = \frac{1}{2}$, we see that

$$x, y \in S \Rightarrow \frac{x+y}{2} \in S.$$

Theorem 8.15. A closed convex subset C of a Hilbert space H contains a unique vector of smallest norm.

Proof. We know that being convex C is non empty and $x,y \in C \Rightarrow \frac{x+y}{2} \in C$.

Let $d = Inf\{\|x\|, x \in C\}$. There exists a sequence $\{x_n\}$ of vectors such that $\|x_n\| \to d$. By the convexity of C, $\frac{x_n+x_n}{2}$ is in C. $\left\|\frac{x_n+x_n}{2}\right\| \geq d$ so $\|x_n+x_n\| \geq 2d$. By $\|gm$ Law, we have

$$\begin{aligned}
\|x_n+x_n\|^2+\|x_m+x_n\|^2 &= 2\|x_m\|^2+2\|x_n\|^2 \\
\Rightarrow \qquad \|x_m+x_n\|^2 &= 2\|x_m\|^2+2\|x_n\|^2-\|x_m+x_n\|^2 \\
&\leq \|x_m\|^2+2\|x_n\|^2-4d^2 \\
&\to 2d^2+2d^2-4d^2 = 0\,[\|x_n\| \to d] \text{ as } m,n \to \infty.
\end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in C. Since H is complete and C is closed; C is complete and their exists a vector x in C such that $x_n \to x$. It is clear by the fact that

$$\|x\| = \|\lim x_n\| = \|d\| = d$$

That x is a vector in C with smallest norm. To see that x is unique, suppose that x' is a vector in C other than x which also has norm d. Then $\frac{x+x'}{2}$ is also in C and we have by $\|gm$ law

$$\begin{aligned}
\left\|\frac{x+x'}{2}\right\|^2 &= \frac{\|x\|^2}{2}+\frac{\|x'\|^2}{2}-\left\|\frac{x+x'}{2}\right\|^2 \\
&< \frac{\|x\|^2}{2}+\frac{\|x'\|^2}{2} = d^2
\end{aligned}$$

which contradicts the definition of d.

Orthogonal Complements

Definition 8.16. Two vectors x and y in a Hilbert space H are said to be orthogonal if

$$(x,y) = 0$$

Since $\overline{(x,y)} = (y,x)$ we have $x \perp y \Leftrightarrow y \perp x$ It is also clear that $x \perp 0$ for every x. Moreover since $(x,x) = \|x\|^2$, 0 is the only vector orthogonal to itself, if $x \perp y, then\, \|x+y\|^2 = \|x-y\|^2 = \|x\|^2+\|y\|^2$ (This is known as Pythagorean theorem).

Definition 8.17. A vector x is said to be orthogonal to a non empty set S (written as $x \perp S$) if $x \perp y$ for every $y \in S$.

Definition 8.18. The set of all vectors orthogonal to S is called orthogonal complement of S and is denoted by $S^{\perp}$.

Theorem 8.19. Let M be a closed linear subspace of a Hilbert space H, let $x \notin M$, and let d be the distance from x to M. Then there exists a unique vector y_0 in M such that

$$\|x - y_0\| = d.$$

Proof. Let M be a closed linear subspace of H, xM and d be the distance from x to. M Then

$$\inf\{\|x-y\|\,; y \in M\}$$

Select a sequence $\{y_n\}$ in M such that $\underset{n\to\infty}{Lnf}\|x_n - y_n\| = d$. Then by parallelogram law

$$\begin{aligned}
\|y_m - y_n\|^2 &= \|(y_m - x) - (y_n - x)\|^2 \\
&= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|(y_m - x) - (y_n - x)\|^2 \\
&= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - \|y_m + y_n - 2x\|^2 \\
&= 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4\left\|\frac{y_m + y_n}{2} - x\right\|^2.
\end{aligned}$$

Since $\frac{y_m+y_n}{2} \in M$, we have

$$\left\|\frac{y_m + y_n}{2} - x\right\| \geq d.$$

Therefore

$$\begin{aligned}
\|y_m - y_n\|^2 &\leq 2\|y_m - x\|^2 + 2\|y_n - x\|^2 - 4d^2 \\
&\to 2d^2 + 2d^2 - 4d^2 = 0, \quad m, n \to \infty.
\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in a closed linear space of a complete space H.

Therefore $\exists$ an element $y_0 \in M$ such that

$$y_0 = \lim_{n\to\infty} y_n.$$

Also

$$d = \lim_{n\to\infty} \|x - y_n\|$$

$$= \|x - \lim y_n\|$$
$$= \|x - y_0\|$$

Uniqueness of y_0. Suppose y_1 and y_2 are two vectors in M such that $\|x - x_1\| = d$ and $\|x - x_2\| = d$. Then to show that $y_1 = y_2$.

Since M is a subspace of H, therefore

$$y_1, y_2 \in M \Rightarrow \frac{(y_1 + y_2)}{2} \in M.$$

Hence by the definition of d, we have

$$\left\| x - \frac{y_1 + y_2}{2} \right\| \geq d \text{ so that } \|2x - (y_1 + y_2)\| \geq 2d.$$

By parallelogram Law, we have

$$\begin{aligned} \|(x - y_1) - (x - y_2)\|^2 &= 2\|x - y_1\|^2 + 2\|x - y_2\|^2 - \|(x - y_1) - (x - y_2)\|^2 \\ \Rightarrow \qquad \|y_2 - y_1\|^2 &= 2\|x - y_1\|^2 + 2\|x - y_2\|^2 + 2\|x - y_2\|^2 - \|2x - (y_1 + y_2)\|^2 \\ &\leq 2d^2 + 2d^2 - 4d^2 = 0 \end{aligned}$$

Thus $\|y_2 - y_2\|^2 \leq 0$ but $\|y_2 - y_2\|^2 \leq 0$.

$$\Rightarrow \|y_2 - y_1\|^2 = 0 \Rightarrow y_2 - y_1 = 0 \Rightarrow y_1 = y_2.$$

Theorem 8.20. If M is a proper closed linear subspace of a Hilbert space H, then there exists a non zero vector z_0 in H such that $z_0 \perp M$.

Proof. Since M is a proper linear subspace of H, then there is a vector x in H which does not belong to M. Let d be distance from x to M. Then (by the above theorem) there exists a vector y_0 in M such that

$$\|x - y_0\| = d.$$

Define $z_0 = x - y_0$.

Since $d > 0, z_0$ is a non-zero vector, we shall show that $z_0 \perp M$. It is sufficient to show that if y is an arbitrary vector in M.

Then $z_0 \perp y$.

For any scalar, we have

$$
\begin{aligned}
& \|z_0-\alpha y\| = \|x-(y_0+\alpha y)\| \geq d = \|z_0\| \\
\Rightarrow\quad & \|z_0-\alpha y\|^2 - \|z_0\|^2 \geq 0 \\
\Rightarrow\quad & (z_0-\alpha y, z_0-\alpha y) - \|z_0\|^2 \geq 0 \\
\Rightarrow\quad & (z_0,z_0) - \overline{\alpha}\,(z_0,y) - \alpha\,(y,z_0) + \alpha\overline{\alpha}\,(y,y) - \|z_0\|^2 \geq 0 \\
\Rightarrow\quad & \|z_0\|^2 - \overline{\alpha}\,(z_0,y) - \alpha\,(y,z_0) + |\alpha|^2\,\|y\|^2 \geq 0 \\
\Rightarrow\quad & -\overline{\alpha}\,(z_0,y) - \alpha\,(y,z_0) + |\alpha|^2\,\|y\|^2 \geq -\|z_0\|^2 \qquad (1)
\end{aligned}
$$

Set $\alpha = \beta\,(z_0,y)$ for an arbitrary real number β. Then (1) becomes

$$-2\beta\,|(z_0,y)|^2 + \beta^2\,|(z_0,y)|^2\,\|y\|^2 \geq 0.$$

If we now put $a = |(z_0,y)|^2$ and

$$b = \|y\|^2,$$

we obtain

$$-2\beta a + \beta^2 ab \geq 0$$

i.e.

$$\beta a\,(\beta b - 2) \geq 0 \qquad (2)$$

for all real. However if $a > 0$,, then (2) is obviously false for all sufficient small *positive* β. We see from this that $a = 0$ i.e. $(z_0,y) = 0$ which implies that $z_0 \perp y$ Hence the theorem.

Theorem 8.21. If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M+N$ is also closed.

Proof. Let z be a limit point of $M+N$. It suffices to show that $z \in M+N$. Let $< z_n >$ be a sequence of points in $M+N$ such that $z_n \to z$. By the assumption that $M \perp N$, we see that M and N are disjoint, so each z_n can be written uniquely in the form $z_n = x_n + y_n$, where $x_n \in$ M and $y_n \in N$. For each $\in > 0$, there exists *apositive* integer N such that

$$
\begin{aligned}
& \|z_m - z_n\| < \in \ \forall\ m,n \geq N(\in) \\
\Rightarrow\quad & \|z_m - z_n\|^2 < \in^2
\end{aligned}
$$

$$\Rightarrow \quad \|(x_m + y_m - (x_n - x_n)\|^2 < \in^2$$
$$\Rightarrow \quad \|(x_m - x_n) + (x_n - y_n)\|^2 < \in^2$$
$$\Rightarrow \quad \|x_m - x_n\|^2 + \|x_m - x_n\| < \in^2$$
$$\Rightarrow \quad \|x_m - x_n\| < \in, \quad \|x_m - x_n\| < \in .$$

Thus $< x_n >$ and $< y_n >$ are Cauchy sequences.

But M and N are closed linear subspaces of x H and therefore, complete. Hence there exists vectors x and y in M and N respectively such that

$x_n \to x.$ and $y_n \to y$ Then

$$z = \lim z_n = \lim (x_n + y_n) = \lim x_n + \lim y_n = x + y \in M + N.$$

Thus every limit point of $M+N$ is in and hence $M+N$ is also closed.

Projection Theorem

Theorem 8.22. If M is a closed linear subspace of a Hilbert space H, then

$H = M \oplus M^{\perp}$, where $M^{\perp}$ = The set of all vectors orthogonal to M.

Proof. Since M and $M^{\perp}$ are orthogonal closed linear subspaces of H, by the previous Theorem, $M + M^{\perp}$ is also a closed linear subspace of H. Moreover, since $M \perp M^{\perp}$, we have $M \cap M^{\perp} = \{0\}$. So it is sufficient to show that $H = M + M^{\perp}$. If this is not so, then $M + M^{\perp}$ is a proper closed linear subspace of H and therefore $\exists$ a vector $z_0 \neq 0$ such that $z_0 \perp \left(M + M^{\perp}\right)$ which is possible only when $z_0 \perp M$ and $z_0 \perp \left(M + M^{\perp}\right)$ that is when $z_0 \perp M$ and $z_0 \in M^{\perp\perp}$ that is when $z_0 \in M^{\perp} \cap M^{\perp\perp}$. But this is impossible since $M^{\perp} \cap M^{\perp\perp} = \{0\}$. Hence $H = M + M^{\perp}$.

Definition 8.23. A non empty subset $\{e_1, e_2, \ldots, e_n, \ldots\}$ of H is called **orthonormal**.

If $\left(e_i, e_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$ Kronoecker Delta $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

Thus orthonormal set consists of mutually orthogonal unit vectors [$\|e_i\| = 1$ for every i].

If H contains only the zero vector, then it has no orthonormal sets. If H contains a non-zero vector x and if we normalize x by considering $e = \frac{x}{\|x\|}$, then the single element set $\{e\}$ is clearly an orthonormal set. In general if $\{x_i\}$ is a non empty set of orthogonal non-zero vector in H and if x_i 's are normalized by replacing each of them by $e_i = \frac{x_i}{\|x_i\|}$, Then the resulting set $\{e_i\}$ is an orthonormal set. If should be noted that if $< x_i >$ is a non-empty set of mutually orthogonal non-zero vectors in H and if in this set, each x_i is replaced by the corresponding unit vector $e_i = \frac{x_i}{\|x_i\|}$, then the resulting set $\{e_i\}$ is an orthonormal set.

Example 8.24. The subset $\{e_1, e_2, \ldots, e_n,\}$ of l_2^n where e_i is the n-tuple with 1 in the ith place and 0's elsewhere, then $\{e_1, e_2, \ldots, e_n,\}$ is an orthonormal set in this space.

Example 8.25. If $\{e_n\}$ is a sequence with 1 in the nth place, and zero elsewhere, then $\{e_1, e_2, \ldots, e_n,\}$ is an orthonormal set in l_2^n.

Theorem 8.26. Let $\{e_1, e_2, \ldots, e_n\}$ be a finite orthonormal set in a Hilbert space H, then

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2 \tag{1}$$

and further

$$x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j \tag{2}$$

Proof. The inequality (1) follows from the following computation.

$$\begin{aligned}
0 &\leq \left\| x - \sum_{i=1}^{n} (x, e_i)\, e_i \right\|^2 \\
&= \left(x - \sum_{i=1}^{n} (x, e_i)\, e_i, \quad x - \sum_{i=1}^{n} \left(x, e_j\right) e_j \right) \\
&= \left(x, x - \sum_{j=1}^{n} (x, e_i)\; e_j \right) - \sum_{i=1}^{n} (x, e_i)\, (e_i, x) - \sum_{j=1}^{n} \left(e_i, e_j\right)\, e_j \Big) \Big) \\
&= \left(x, x - \sum_{j=1}^{n} \overline{(x, e_j)}\, \left(x, e_j\right) - \sum_{i=1}^{n} (x, e_i) \left[\left(e_i, x\right) - \sum_{j=1}^{n} \overline{(x, e_j)}\, \left(x, e_j\right) \right] \right. \\
&= (x, x) - \sum_{j=1}^{n} \overline{(x, e_j)}\, \left(x, e_j\right) - \sum_{i=1}^{n} (x, e_i)\, (e_i, x) \\
&\quad + \sum_{i=1}^{n} \sum_{i=1}^{n} (x, e_i)\, \overline{(x, e_j)}\, \left(e_i, e_j\right)
\end{aligned}$$

$$= \|x\|^2 - \sum_{i=1}^{n} (x, e_i)\,\overline{(x, e_i)} - \sum_{j=1}^{n} \left(x, e_j\right) \overline{\left(x, e_j\right)}$$
$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} (x, e_i)\,\overline{\left(x, e_j\right)} \left(e_i, e_j\right)$$
$$= \|x\|^2 - \sum_{i=1}^{n} |(x, e_i)|^2$$

$$\Rightarrow \quad \sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

Also we observe that

$$\left(x - \sum_{i=1}^{n} (x, e_i)\, e_i, e_j\right) = \left(x, e_j\right) - \sum_{i=1}^{n} (x, e_i) \left(e_i, e_j\right)$$
$$= \left(x, e_j\right) - \left(x, e_j\right)$$
$$= 0.$$

Hence

$$x - \sum_{i=1}^{n} (x, e_i)\, e_i \perp e_j \text{ for each } j.$$

Inequality (1) is called the special case of a more general inequality known as Bessel's inequality.

Theorem 8.27. If $< e_i >$ is an orthonormal set in a Hilbert space H and if x is any vector in H, then the set $S = \{e_i; (x, e_i) \neq 0\}$ is either empty or countable.

Proof. For each positive integer n, consider the set

$$S_n = \left\{ e_i; |(x, e_i)|^2 > \frac{\|x\|^2}{n} \right\}$$

S_n can not contain more than $n-1$ vectors, since in that case $\sum_{i=1}^{p} |x, e_i|^2 > \|x\|^2$ when $p > (n-1)$ and thus contradicts the above theorem. Also, each member of S is contained in $\bigcup_{n=1}^{\infty} S_n$. But union of a countable collection of countable sets is countable. Therefore $\bigcup_{n=1}^{\infty} S_n$ and hence S is countable.

Theorem 8.28. If $< e_i >$ is an orthonormal set in a Hilbert space H, then

$$\sum |(x,e_i)|^2 \leq \|x\|^2$$

for every vector $x \in H$.

Proof. Let $S = \{e_i; (x,e_i) \neq 0\}$. If S is empty, then we define $\sum |(x,e_i)|^2$ to be the number zero and the result is obvious in this case. We now assume that S is non-empty. Then by the above theorem, it must be finite or countably infinite. If S is finite, then it can be written in the form

$$S = \{e_1, e_2, \dots e_n\}$$

for some $+ve$ integer n. In this case, we define $\sum |(x,e_i)|^2$ to be $\sum |(x,e_i)|^2$. The inequality to be proved now reduce to

$$\sum_{i=1}^{n} |(x,e_i)|^2 \leq \|x\|^2$$

which has already been proved.

Now consider the case

$$S = [e_i, (x,e_i) \neq 0]$$

is countably infinite.

Let the vectors in S be arranged in a definite order.

$$S = \{e_1, e_2, \dots, e_n, \dots\}$$

By the theory of absolutely convergent series, if $\sum_{n=1}^{\infty} |(x,e_n)|^2$ converges, then every series obtained from it by rearranging its terms and also converges and all such series have the same sum. We, therefore, define $\sum |(x,e_i)|^2$ to be $\sum_{n=1}^{\infty} |(x,e_n)|^2$ and it follows from the above remark that $\sum_{n=1}^{\infty} |(x,e_n)|^2$ is a non-negative extended real number which depends only on S and not on the arrangement of its vectors. We now observe that

$$\begin{aligned}
\sum |(x,e_i)|^2 &= \sum_{i=1}^{n} |(x,e_i)|^2 \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} |(x,e_i)|^2 \\
&= \lim_{n\to\infty} \|x\|^2 = \|x\|^2
\end{aligned}$$

Hence
$$\sum_{n=1}^{\infty} |(x,e_n)|^2 \leq \|x\|^2 \text{ for every } x \in H.$$

Theorem 8.29. If $< e_i >$ is an orthonormal set in a Hilbert space H, and if x is any vector in H, then
$$x - \sum (x,e_i)\, e_i \perp e_j$$
for each j.

Proof. We set
$$S = \{e_i, (x,e_i) \neq 0\}$$
when S is empty, we define $\sum_{i=1}^{n} (x,e_i)\, e_i$ to be the vector zero and then the required result reduces to the statement that $x - 0 = x$ is orthogonal to each e_j, which is precisely, what is meant by saying that S is empty. When S is non-empty and finite, then it can be written in the form.
$$S = < e_1, e_2, \ldots, e_n >$$
and we define $\sum (x,e_i)\, e_i$ to be $\sum (x,e_i)\, e_i$ and in that case the required result reduces to $x - \sum_{i=1}^{n} (x,e_i)\, e_i \perp e_j$ which has already been proved.

We may assume for the remainder of proof that S is countably infinite. Let the vectors in S be listed in a definite order $S = < e_1, e_2, \ldots, e_n, \ldots >$. We put $S_n - \sum_{i=1}^{n} (x,e_i)\, e_i$ and we note that for $m > n$, we have
$$\|S_m - S_n\|^2 = \left\| \sum_{i=n+1}^{m} (x,e_i)\, e_i \right\|^2 \sum_{i=n+1}^{m} |(x,e_i)|^2 \leq \|x\|^2 .$$

Bessel's inequality shows that the series $\sum_{n=1}^{\infty} |(x,e_n)|^2$ converges and so $< S_n >$ is a Cauchy in H and since H is complete, this sequence converges to a vector S, which we write in the form $S = \sum_{n=1}^{\infty} (x,e_n)\, e_n$.

We now define $\sum_{i=1}^{n} (x,e_i)\, e_i$ to be $\sum_{n=1}^{\infty} (x,e_n)\, e_n$ (without considering the effect of rearrangement) and observe that the required result follows from $x - \sum_{i=1}^{n} (x,e_i)\, e_i \perp e_j$ and the continuing of the inner product.

$$\begin{aligned}
\left(x - \sum_{i=1}^{n} (x,e_i)\, e_i, e_j \right) &= \left(x - S, e_j \right) \\
&= \left(x, e_j \right) - \left(S, e_j \right)
\end{aligned}$$

$$\begin{aligned} &= (x, e_j) - \left(\lim_{n\to\infty} S_n, e_j\right) \\ &= (x, e_j) - \lim_{n\to\infty} (S_n, e_j) \\ &= (x, e_j) - (x, e_j) = 0. \end{aligned}$$

All that remains to show that this definition of $\sum (x, e_i) e_i$ is valid in the sense that it does not depend on the arrangement of vectors in S. Let the vectors in S be rearranged in any manner;

$$S = \{f_1, f_2, \ldots, f_n, \ldots, \}$$

We put $S'_n = \sum_{i=1}^{n} (x, f_i) f_i$ and we see as above that the sequence $< f_n >$ converges to the limit S', which we write in the form $S' = \sum_{n=1}^{\infty} (x, f_n) f_n$ We conclude the proof by showing that S' equals S. Let $\in > 0$ be given and let n_0 be +ve integer so large that if $n \geq n_0$, then $\|S_n - S\| < \in$, and $\|S'_n - S'\| < \in$ and $\sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \in^2$. For some *positive* integer $m_0 > n_0$, all terms of S_{no} occure among those of S'_{m_0} so $S'_{m_0} - S'_{n_0}$ is a finite sum of terms of the form $(x, e_i) e_i$ for $e = n_0 + 1, n_0 + 2, \ldots$ This yields

$$\left\|S'_{m_0} - S'_{n_0}\right\|^2 \leq \sum_{i=n_0+1}^{\infty} |(x, e_i)|^2 < \in^2 .$$

So

$$\left\|S'_{m_0} - S'_{n_0}\right\| < \in$$

and

$$\left\|S' - S\right\| \leq \left\|S' - S'_{m_0}\right\| + \left\|S'_{m_0} - S'_{n_0}\right\| + \|S_{n_0} - S\| < \in + \in + \in = 3 \in$$

Since $\in$ is arbitrary, this shows that $S' = S$.

Definition 8.30. An orthonormal set $E = \{e_i\}$ in a Hilbert space H is said to be complete if the only vector orthogonal to all elements of E is zero. Thus an orthonormal set $< e_i >$ is complete if there does not exist a single vector which is orthogonal to all vectors in E, unless the vector is zero. That is, if it is not possible to adjoin a vector e to $< e_i >$ in such a way that $< e_i, e >$ is an orthonormal set which properly contains $< e_i >$.

Theorem 8.31. Every non-zero Hilbert space contains a complete orthonormal set.

Proof. Let H be a non-zero Hilbert space and $x \in H, x \neq 0$. Normalize x by writing $e = \frac{x}{\|x\|}$, then clearly $< e >$ is an orthonormal set. It follows therefore that every non-zero Hilbert space surely contains orthonormal sets. Consider the collection of all possible orthonormal sets in H, then the collection has a maximal member M since by Zorn's lemma, if P is partially ordered set in which every chain has an upper bound, then P possesses a maximal element, we shall show that M is complete. Suppose that $y \neq 0$ and $y \perp M$ then put

$$z = \frac{y}{\|y\|}$$

we observe $M \cup < z >$ that is also an orthonormal set and thus contradicts the maximality of M. Hence $y \perp M$ only if $y = 0$.

Theorem 8.32. Let H be a Hilbert space and let $< e_i >$ be an orthonormal set in H.

Then the following conditions are all equivalent to one another:

(1) $< e_i >$ is complete

(2) $x \perp < e_i > \Rightarrow x = 0$.

(3) If x is any arbitrary vector in H, then $x = \sum (x, e_i)\, e_i$.

(4) If x is any arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$

Proof. (1)⇒(2): Let $< e_i >$ be complete, if (2) is not zero, then $\exists$ a vector $x \neq 0$, such that $x \perp < e_i >$. Define $e = \frac{x}{\|x\|}$ then the vector e (is a unit vector and) is orthogonal to each member of $< e_i >$. Hence the set obtained by joining e to $< e_i >$ becomes an orthonormal set containing $< e_i >$. This contradicts the completeness of $< e_i >$. Hence $\perp < e_i > \Rightarrow x = 0$.

(2)⇒(3): Suppose that $x \perp < e_i > \Rightarrow x = 0$. Let x be an arbitrary element in H, then $x - \sum (x, e_i)\, e_i$ is orthogonal to each e_j for all j and therefore to $< e_i >$. Therefore (2) implies that

$$\begin{aligned} & x - \sum (x, e_i)\, e_i = 0 \\ \Rightarrow \quad & x = \sum (x, e_i)\, e_i \end{aligned}$$

(3)⇒(4). Suppose that x is an arbitrary vector in H such that. $x - \sum (x, e_i)\, e_i$

Then by inner product, we have

$$\|x^2\| = (x, x) = \left(\sum_i (x, e_i)\, e_i, \sum_j (x, e_j)\, e_j \right)$$

$$= \sum_i (x, e_i) \left\{ \sum_j \overline{(x, e_i)} \right\} (e_i, e_j)$$
$$= \sum_i (x, e_i) \overline{(x, e_i)}$$
$$= \sum_i |(x, e_i)|^2 .$$

(4)⇒(1): We are given that if x is an arbitrary vector in H, then $\|x\|^2 = \sum |(x, e_i)|^2$. Suppose that $< e_i >$ is not complete, then it is a proper subset of an orthonormal set $< e_i, e >$. Since e is orthogonal to all e_i's such that $\|e\| = 1$, we have

$$\|e\|^2 = \sum |(e, e_i)|^2 = 0 \Rightarrow \quad e = 0$$

this contradicts the fact that e is a unit vector. Hence $< e_i >$ is complete.

Remark 8.33. If $< e_i >$ is a complete orthonormal set in a Hilbert space H and let x be an arbitrary vector in H, then the numbers $< x, e_i >$ are called Fourier coefficients of x, the expression $x = \sum (x, e_i) e_i$ is called the Fourier expansion of x and equation $\|x\|^2 = \sum |(x, e_i)|^2$ is called **Parseval's equation**.

Example 8.34. Consider the Hilbert space $L_2 [0, 2\pi]$. This space consists of all complex functions defined on $[0, 2\pi]$ which are Lebesgue measurable and square integrable in the sense that

$$\int_0^{2\pi} |f(x)|^2 dx < \infty.$$

Norm and Inner product in $L_2 (0, 2\pi)$ are defined by

$$\|f\| = \left(\int_0^{2\pi} |f(x)|^2 dx \right)^{1/2}$$

$$(f, g) = \int_0^{2\pi} f(x) . \overline{g(x)} dx$$

A simple computation shows that the function e^{inx} for $n = 0, \pm 1, \pm 2, \ldots$ are mutually orthogonal in L_2,

$$\int_0^{2\pi} e^{inx} e^{-inx} dx = \begin{cases} 0, & m \neq n \\ 2\pi & \text{if } m \neq n \end{cases}$$

It follows from this that the functions $e_n\,(n=0,\pm1,\pm2,\ldots)$ defined by $e_n\,(x)=e^{inx}/\sqrt{2\pi}$ from an orthonormal set in L_2. For any function f in L_2, the numbers

$$C_n=(f,e_n)=\frac{1}{\sqrt{2\pi}}\int_0^{2\pi} f\,(x)\,e^{-inx}dx \tag{1}$$

are its classical Fourier coefficients and Bessel's inequality takes the form.

$$\sum_{n=-\infty}^{\infty}|C_n|^2\leq\int_0^{2\pi}|f\,(x)|^2\,dx.<\infty.$$

It is a fact of very great importance in the theory of Fourier series that the orthonormal set $<e_n>$ is complete in L_2. As we have seen that for every f in L_2, Bessel's inequality can be strengthened to Parseval's equation:

$$\sum_{n=-\infty}^{\infty}|C_n|^2=\int_0^{2\pi}|f\,(x)|^2\,dx.$$

The previous theorem also tells us that the completeness of $<e_n>$ is equivalent to the statement that each f in L_2 has a Fourier expansion

$$f\,(x)=\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty}C_ne^{-inx}.$$

Gram-Schmide Orthogonalization Process

Suppose that $<x_1,x_2\ldots,x_n,\ldots>$ is a linearly independent set in a Hilbert space H. Our aim is to convert it into the corresponding orthonormal set $<e_1,e_2,\ldots,e_n,\ldots>$ with the property that for each n, the linear subspace of H is spanned by $<e_1,e_2,\ldots,e_n,\ldots>$

Our first step is to normalize x_1 by putting

$$e_1=\frac{x_1}{\|x_1\|}$$

Let us consider $x_2-(x_2,e_1)\,e_1$. It is orthogonal to e_1 and we normalize this by putting

$$e_2=\frac{x_2-(x_2,e_1)\,e_1}{\|x_2-(x_2,e_1)\,e_1\|}$$

Now e_1 and e_2 are orthogonal. Consider $x_3-(x_3,e_1)\,e_1-(x_3,e_2)\,e_2$.. It is orthogonal to e_1 and e_2. We normalize it by

$$e_2=\frac{x_3-(x_3,e_1)\,e_1-(x_3,e_2)\,e_2}{\|x_3-(x_3,e_1)\,e_1-(x_3,e_2)\,e_2\|}$$

We see that $(x_3,e_1)-0\,(x_3,e_2)=0$. Continuing this process, we obtain an orthonormal set $<e_1,e_2,\ldots,e_n,\ldots>$ with the required properties.

CHAPTER 9

Conjugate of Hilbert Spaces

In the chapter 4 and 5, we have studied conjugate of sequence spaces and obtained series representation of some conjugate spaces like l_p space. We have also explained that we can not obtain the conjugate space of l_∞ with the tool developed in text. But in case of general Banach space the conjugate space of Banach space is different Banach space with different norm. The most surprising fact about conjugate space of Hilbert space H is that the conjugate space H^* of H is in some sense is the same as H itself. The theorem identifying H with H^* is known as Riesz-Representation theorem for continuous linear functional on H, l_2 serves as an instance of this theorem. Futher in present chapter a correspondance between H and H^* is established. Also we prove Riesz-Representation theorem for continuous linear functional on H. Finally we show that H^* is itself a Hilbert space and H is reflexive.

Theorem 9.1. Let y be a fixed vector in Hilbert Space H and let f_y be a function defined as $f_y(x) = (x,y)$ for every $x \in H$. Then f_y is a functional on H and $\|y\| = \|f_y\|$.

Proof. Let H be a Hilbert space and H^* its conjugate space. Let y be a fixed vector in H, Define a function f_y on H by

$$f_y(x) = (x,y), \forall x \in H.$$

We assert that f_y is linear, for

$$\begin{aligned} f_y(x_1+x_2) &= (x_1+x_2,y) \quad \forall\, x_1+x_2 \in H \\ &= (x_1+x) + (x_2,y) \\ &= f_y(x_1) + f_y(x_2,) \end{aligned}$$

and

$$\begin{aligned} f_y(\alpha x) &= (\alpha x, y) \\ &= \alpha(x,y) = \alpha(f_y(x)) \end{aligned}$$

Also

$$|f_y(x)| = |(x,y)| \leq \|x\|.\|y\| \quad \text{(By Schwartz's Inequality)}$$

which proves that

$$\left\|f_y\right\| \leq \|y\| \tag{1}$$

which implies that f_y is continuous Thus f_y is linear and continuous mapping and hence is a linear functional on H. On the other hand if $y = 0$, then

$$f_y(x) = (x, 0) = 0 \Rightarrow \left\|f_y\right\| = \|y\| = 0.$$

If $y \neq 0$, then

$$\begin{aligned}
\left\|f_y\right\| &= \sup\left\{\left|f_y(x)\right|; \|x\| = 1\right\} \\
&\geq \left|f_y\left(\frac{y}{\|y\|}\right)\right| \quad \{Beacuse||\frac{y}{||y||}|| = 1\} \\
&= \left|\left(\frac{y}{\|y\|}\right), y\right|
\end{aligned}$$

$$\Rightarrow ||f_y|| \geq ||y|| \tag{2}$$

Hence from (1) and (2), we have

$$\left\|f_y\right\| = \|y\|$$

Thus for each $y \in H$. There is a linear functional $f_y \in H^*$ such that

$$\left\|f_y\right\| = \|y\|.$$

Hence the mapping $y \to f_y$ is a norm preserving mapping of H into H^*,

Riesz-Representation Theorem for Hilbert spaces

Theorem 9.2. Let H be a Hilbert space and let f be an arbitrary functional in H^*. Then there exists a unique vector y in H such that $f(x) = (x, y)$ for every x in H.

Proof. We shall show first that if such a y exists, then it is necessarily unique. Let y' be another vector in H such that $f(x) = (x, y')$. Then clearly $(x, y) = (x, y')$ i.e. $(x, y - y') = 0$ for all x in H. Since zero is the only vector orthogonal to every vector, this implies that $y - y' = 0$ which implies that $y' = y$.

Now we turn to the existence of such vector y. If $f = 0$, then it clearly suffices to choose $y = 0$. We may therefore assume that $f \neq 0$. The null space $M =$

$\{x \in H; f(x) = 0\}$ is thus a proper closed linear subspace of H and therefore there exists a non-zero vector y_0 in H which is orthogonal to M. We show that if is a suitably chosen scalar, then the vector $y = \alpha y_0$ meets our requirements. If $x \in M$, then whatever values of may be, we have

$$f(x) = (x, \alpha y_0) = 0.$$

We now choose $x = y_0$. Then we must have

$$f(y_0) = (y_0, \alpha y_0) = \overline{\alpha}(y_0, y_0) = \overline{\alpha}\|y_0\|^2.$$

and therefore we must choose our scalar α such that

$$\overline{\alpha} = \frac{f(y_0)}{\|y\|^2} \quad \text{or} \quad \alpha = \frac{\overline{f(y_0)}}{\|y\|^2}$$

therefore it follows that the vector $\alpha y_0 = \frac{\overline{f(y_0)}}{\|y\|^2}.y_0$ satisfies the required condition for each $x \in M$ and for $x = y_0$. Each x in H can be written in the form $x = m + \beta y_0, m \in M$. For this all that is necessary is to choose β in such a way that $f(x - \beta y_0) = f(x) - \beta f(y_0) = 0$ and this is justified by putting $\beta = \frac{f(x)}{f(y_0)}$.

Now we show that the conclusion of the theorem holds for each x in H. For this, we have

$$\begin{aligned} f(x) &= f(m + \beta y_0) = f(m) + \beta f(y_0) \\ &= (m, y) + \beta(y_0, y) \\ &= \beta(m + \beta y_0, y) = (x, y) \end{aligned}$$

Remark 9.3. It follows from this theorem that the norm preserving mapping of H into H^* defined by $y \to f_y$ where $f_y(x) = (x, y)$ is actually a mapping of H onto H^*.

Remark 9.4. It would be pleasant if $y \to f_y$ were also a linear mapping. This is not quite true, however, for

$$f_{y1} + f_{y2} = f_{y1} + f_{y2} \quad \text{and} \quad f_{\alpha y} = \overline{\alpha} f_y \tag{1}$$

Also it follows from (1), that the mapping $y \to f_y$ is an isometry, for

$$\left\|f_x - f_y\right\| = \left\|f_{x-y}\right\| = \|x - y\|.$$

The Adjoint of an operator

Let y be a vector in a Hilbert space H and f_y its corresponding functional in H^*.

Operate with T^* on f_y to obtain a functional $f_z = T^* f_y$ and return to its corresponding vector z in H. There are three mappings under consideration here $(H \to H^* \to H^* \to H)$ and we are forming their product:

$$y \to f_y \to T^* f_y = f_z \to z \tag{1}$$

An operator T^* defined on H by

$$T^*(y) = z$$

is called adjoint of operator T.

The same symbol is used for the adjoint of T as for its conjugate because these two mappings are actually the same if H and H^* are identified by means of natural correspondence. It is easy to keep track of whether T^* signifies the conjugate or the adjoint of T by noticing whether it operates on functionals or on vectors.

Let x be an arbitrary vector in H. Then we have

$$(T^* f_y)(x) = f_y(T(x)) = (T(x), y)$$

and

$$(T^* f_y)(x) = f_z(x) = (x, T^* y)$$

so that

$$(Tx, y) = (x, T^* y) \text{ for all } x \text{ and } y.$$

The adjoint of an operator T is unique, for let T be another operator on H. such that

$$\begin{aligned}
& (Tx, y) = (x, T^* y) \quad \text{for all } x, y \in H. \\
\Rightarrow\quad & (x, T^* y) = \left(x, T' y\right) \\
\Rightarrow\quad & (x, T^* y - T^* y) = 0. \\
\Rightarrow\quad & T^* y - T' y = 0 \\
\Rightarrow\quad & T^* y = T' y \quad \forall\, y \in H. \\
\Rightarrow\quad & T^* = T'
\end{aligned}$$

We now prove that T^* actually is an operator on H (all we know so far is that it maps H into itself) for any y and z and for all x in H, we have

$$(x, T^*(\alpha y + \beta z)) = (Tx, \alpha y + \beta z)$$

$$
\begin{aligned}
&= \overline{\alpha}\,(Tx,y) + \overline{\beta}\,(Tx,y) \\
&= \overline{\alpha}\,(x,T^*y) + \overline{\beta}\,(x,T^*y) \\
&= (x,\alpha T^*y) + (x,\beta T^*y) \\
&= (x,\alpha T^*y + \beta T^*z)
\end{aligned}
$$

Hence T^* is linear. It remains to show that T^* is cont. To prove this, we note that

$$
\begin{aligned}
\|T^*y\|^2 &= (T^*y,T^*y) = (TT^*y,y) \\
&\leq \|TT^*y\| \;\; \|y\| \\
&\leq \|T\|\,\|T^*y\|\,\|y\|
\end{aligned}
$$

which implies that $\|T^*y\| \leq \|T\|\,\|y\|$ for all y and therefore

$$\|T^*\| \leq \|T\|$$

Hence T^* is continuous. It follows therefore that $T \to T^*$ is a mapping of $\beta\,(H)$ into itself. This mapping is called the **adjoint operator**on $\beta\,(H)$.

Theorem 9.5. The adjoint operator $T \to T^*$ on $\beta\,(H)$ has the following properties:

(1) $(T_1+T_2)^* = T_1^* + T_2^*$

(2) $(\alpha T)^* = \overline{\alpha}T^*$

(3) $(T_1T_2)^* = T_2^*T_1^*$

(4) $T^{**} = T$

(5) $\|T^*\| = \|T\|$

(6) $\|T^*T\| = \|T\|^2$

for all scalars and $T_1, T, T_2 \in \beta\,(H)$.

Proof. To prove (1), we have

$$
\begin{aligned}
\left(x,(T_1+T_2)^*\,y\right) &= ((T_1+T_2)\,x,y) \\
&= (T_1x+T_2x,y) \\
&= (T_1x,y) + (T_2x,y) \\
&= (x,T_1^*y) + (x,T_2^*y)
\end{aligned}
$$

$$
\begin{aligned}
&= (x, T_1^* y + T_2^* y) \\
&= (x, (T_1^* + T_2^*)\, y) \\
\Rightarrow \qquad (T_1 + T_2)^* &= T_1^* + T_2^*
\end{aligned}
$$

(2) If $x \in H$, then

$$
\begin{aligned}
\big(x, (\alpha T)^*\, y\big) &= (\alpha T x, y) \\
&= \alpha\, (T x, y) = \alpha\, (x, T^* y) \\
&= (x, \overline{\alpha} T^* y) = (x, (\overline{\alpha} T^*)\, y) \\
\Rightarrow \quad (\alpha T)^* &= \overline{\alpha} T^*
\end{aligned}
$$

(3) For all $x, y \in H$, we have

$$
\begin{aligned}
\big(x, (T_1 T_2)^*\, y\big) &= ((T_1 T_2)\, x, y) \\
&= (T_1\, (T_2 x)\, , y) \\
&= (T_2 x, T_1^* y) \\
&= (x T_2^*\, (T_1^* y)) \\
&= (x, (T_2^* T_1^*)\, y)
\end{aligned}
$$

Thus by the uniqueness of adjoint operator.

$$
(T_1 T_2)^* = T_2^* T_1^*
$$

(4) For all $x, y \in H$, we have

$$
\begin{aligned}
(x, T^{**} y) &= \big(x, (T^*)^*\, y\big) \\
&= (T^* x, y) \\
&= \overline{(y, T^* x)} = \overline{(T y, x)} \\
\Rightarrow \qquad T^{**} &= T
\end{aligned}
$$

(5) Let y be an arbitrary vector in H. Then

$$
\begin{aligned}
\|T^* y\|^2 &= (T^* y, T^* y) \\
&= (T T^* y, y) \\
&= |(T T^* y, y)| \\
&\le \|T T^* y\|\, \|y\| \\
&\le \|T\|\, \|T^* y\|\, \|y\|
\end{aligned}
$$

$$\Rightarrow \quad \|T^*y\| \leq \|T\| \, \|y\|$$
$$\Rightarrow \quad \|T^*\| \leq \|T\|$$

Replacing T be T^* in the above inequality, we have

$$\left\|(T^*)^*\right\| \leq \|T^*\|$$
$$\Rightarrow \quad \|T\| \leq \|T^*\|$$

Hence $\|T\| = \|T^*\|$.

(6) To prove this equality, we have

$$\begin{aligned} \|T^*T\| &\leq \|T^*\| \, \|T\| = \|T\| \, \|T\| \qquad \text{[using (5)]} \\ &= \|T\|^2 \end{aligned}$$

and

$$\begin{aligned} \|Tx\|^2 &= (Tx, Tx) = (x, T^*Tx) \\ &\leq \|x\| \, \|T^*Tx\| \\ &\leq \|x\| \, \|T^*T\| \, \|x\| \\ &= \|x\|^2 \, \|T^*T\| \end{aligned}$$

$$\Rightarrow \quad \left\{ \frac{\|Tx\|^2}{\|x\|^2}, \quad x \neq 0 \right\} \leq \|T^*T\|$$
$$\Rightarrow \quad \sup \left\{ \frac{\|Tx\|^2}{\|x\|^2}, x \neq 0 \right\} \leq \|T^*T\|$$

$$\Rightarrow \quad \|T\|^2 \leq \|T^*T\| \tag{2}$$

from (1) and (2)

Example 9.6. Show that adjoint operation is one-one onto as mapping of B(H) into itself.

Solution: Let $\phi : B(H) \to B(H)$ is defined as

$$\phi(T) = T^*, \forall T \in B(H)$$

ϕ **is one-one:** Let $T^1, T^2 \in B(H)$, then

$$
\begin{aligned}
\phi(T_1) = \phi(T_2) \\
&\Rightarrow T_1^* = T_2^* \\
&\Rightarrow (T_1^*)^* = (T_2^*)_* \\
&\Rightarrow T_1^{**} = T_2^{**} \\
&\Rightarrow T_1 = T_2.
\end{aligned}
$$

Hence ϕ is one-one.
ϕ **is onto:** Let $T \in B(H)$ then $T^* \in B(H)$ and we have

$$\phi(T^*) = (T^*)^* = T^{**} = T$$

Hence, the mapping ϕ is onto.

Self-Adjoint Operator

Now we study some special types of operators defined on a Hilbert space. The definitions and properties of these operators depend mostly on the properties of the adjoint of an operator.

Definition 9.7. An operator A on a Hilbert space is said to be self-adjoint if it equals its adjoint i.e. if $A = A^*$.

We know that $0^* = 0$ and $1^* = 1$, so zero and I are self adjoint operator. If is real and A_1 and A_2 are self-adjoint, we claim that $A_1 + A_2$ and αA_1 are also self-adjoint. We establish these facts in the form of a more general theorem:

Theorem 9.8. The self adjoint operators in $B(H)$ form a closed real linear subspace of $B(H)$ and therefore a real Banach space-which contains the identity transformation.

Proof. If A_1 and A_2 are self-adjoint and if α and β are real numbers, then

$$
\begin{aligned}
(\alpha A_1 + \beta A_2)^* &= (\alpha A_1)^* + (\beta A_2)^* \\
&= \overline{\alpha} A_1^* + \overline{\beta} A_2^* \\
&= \alpha A_1^* + \beta A_2^*.
\end{aligned}
$$

[Since α, β are real and $A_1^* = A_1, A_2^* = A_2^*$.

$\Rightarrow \alpha A_1 + \beta A_2$ is also self-adjoint. Therefore set of all self-adjoint operators A in $\beta(H)$ is its linear subspace.

Further, if $< A_n >$ is a sequence of self-adjoint operators which converges to an operator A, then it can be seen that A is also self-adjoint. In fact

$$\begin{aligned}
\|A - A^*\| &= \|A - A_n + A_n - A_n^* + A_n^* - A^*\| \\
&\leq \|A - A_n\| + \|A_n - A_n^*\| + \|A_n^* - A^*\| \\
&= \|A - A_n\| + \left\|(A_n - A_n)^*\right\| \\
&= \|A - A_n\| + \|A_n - A\| \quad [\text{using } \|A^*\| = \|A\|] \\
&= 2\|A_n - A\| \to 0.
\end{aligned}$$
$$\Rightarrow \quad A - A^* = 0 \text{ so } A = A^*.$$

Also $I^* = I$.

Hence the set of all self-adjoint operators in $B(H)$ form a closed linear subspace of $B(H)$ containing identity transformation and therefore is a real Banach space containing the identity transformation.

Theorem 9.9. If A_1 and A_2 are self-adjoint operators on H, then their product

$A_1\, A_2$ is self-adjoint iff $A_1A_2 = A_2A_1$.

Proof. Suppose first that A_1A_2 is self-adjoint, then

$$A_1A_2 = (A_1A_2)^* = A_2^*A_1^* = A_2A_1$$

Conversely suppose that $A_1A_2 = A_2A_1$.. Then

$$(A_1A_2)^* = A_2^*A_1^* = A_2A_1 = A_1A_2$$

and therefore A_1A_2 is self-adjoint.

Theorem 9.10. If T is an arbitrary operator on H, then $T = 0 \Leftrightarrow (Tx, y) = 0$ for all x and y.

Proof. If $T = 0$, then $(Tx, y) = (0x, y) = (0, y) = 0$ for all $x, y \in H$. On the other hand if $(Tx, y) = 0$ for all x and y in H, then in particular $(Tx, Tx) = 0$ for all $x \in H$ which means that $Tx = 0$ for all $x \in H$ and therefore $T = 0$.

Theorem 9.11. If T is an operator on H, then $T = 0$ iff $(Tx, x) = 0$ for all x.

Proof. If $T = 0$, then

$$(Tx,x) = (0x,x) = (0,x) = 0 \quad \forall\, x \in H.$$

Conversely suppose that $(Tx,x) = 0$ for all $x \in H$. We shall show that $T = 0$, which holds if $(Tx,y) = 0$ for all $xy \in H$. So it suffices to prove that $(Tx,y) = 0$ for all $x,y \in H$. The proof of this depends on the following identity.

$$(T(\alpha x + \beta y), \alpha x + \beta y) - |\alpha|^2 (Tx,x) - |\beta|^2 (Ty,y) = \alpha\overline{\beta}\,(Tx,y) + \overline{\alpha}\beta\,(Ty,x) \tag{1}$$

By our hypothesis, the left side of (1) and therefore the right side as well equals zero for all α and β. If we put $\alpha = 1, \beta = 1$ in (1), we get

$$(Tx,y) + (Ty,x) = 0 \tag{2}$$

and if we put $\alpha = i$ and $\beta = 1$, we get

$$i\,(Tx,y) - i\,(Ty,x) = 0$$

and therefore

$$(Tx,y) - (Ty,x) = 0 \tag{3}$$

Adding (1) and (3), we have

$$(Tx,y) = 0 \text{ for all } x,y \in H.$$

Hence $T = 0$.

Theorem 9.12. An operator T on H is self adjoint iff (Tx,x) is real for all x.

Proof. If T is self adjoint, then

$$\overline{(Tx,x)} = (x,Tx) = (x,T^*x) = (Tx,x)$$

shows that (Tx,x) is real for all x, On the hand, if (Tx,x) is real for all x, then

$$\begin{aligned}
& (Tx,x) = \overline{(Tx,x)} = \overline{(x,T^*x)} = (T^*x,x) \\
\Rightarrow\quad & ((T - T^*)x,x) = 0 \\
\Rightarrow\quad & T - T^* = 0 \\
\Rightarrow\quad & T = T^*
\end{aligned}$$

Definition 9.13. If A_1 and A_2 are self-adjoint operators on a Hilbert space H, we write $A_1 \leq A_2$ if $(A_1x,x) \leq (A_2x,x)$ for all $x \in H$.

Theorem 9.14. The real Banach space of all self-adjoint operators on H is a partially ordered set whose linear structure and order structure are related by following properties:

(i) If $A_1 \leq A_2$, then $A_1 + A \leq A_2 + A$ for every A.

(ii) If $A_1 \leq A_2$ and $\alpha \geq 0$, then $A_1 \leq \alpha A_2$.

Proof. Suppose B is the Banach space consisting of all self-adjoint operators on H. We define relation $\leq$ on B by

$$A_1 \leq A_2 \text{ if } (A_1x,x) \leq (A_2x,x) \qquad \forall\, x \in H, A_1, A_2 \in B.$$

Then

(i) $(Ax,x) = (Ax,x)\,\forall x \in H, A \in B$ implies $A \leq A \forall A \in B$. Hence $\leq$ is reflexive.

(ii) If $A_1, A_2 \in B$ such that $A_1 \leq A_2$ and $A_1 \leq A_2$, then

$$A_1 \leq A_2 \Rightarrow (A_1x,x) \leq (A_2x,x)$$
$$A_2 \leq A_1 \Rightarrow (A_2x,x) \leq (A_1x,x)$$

Combining these two expressions, we have

$$\begin{aligned} & (A_1x,x) = (A_2x,x) \\ \Rightarrow \quad & ((A_1 - A_2)x,x) = 0 \Rightarrow A_1 - A_2 = 0 \\ \Rightarrow \quad & A_1 = A_2 \end{aligned}$$

Therefore the relation $\leq$ is anti-symmetric.

(iii) Let $A_1, A_2, A_3 \in B$ such that $A_1 \leq A_2$ and $A_2 \leq A_3$. Then

$$A_1 \leq A_2 \Rightarrow (A_1x,x) \leq (A_2x,x)$$
$$A_2 \leq A_3 \Rightarrow (A_2x,x) \leq (A_3x,x)$$

On both of these yield

$$\begin{aligned} & (A_1x,x) \leq (A_3x,x) \\ \Rightarrow \quad & A_1 \leq A_3\,. \end{aligned}$$

Thus the relation is transitive.

Hence $\leq$ is a partially ordered relation. Now we prove the relation (1) and (2)

$$\begin{aligned} A_1 \leq A_2 &\Rightarrow (A_1x,x) \leq (A_2x,x) \qquad (1)\\ &\Rightarrow (A_1x,x)+(Ax,x) \leq (A_2x,x)\,(Ax,x)\\ &\Rightarrow ((A_1+A)x,x) \leq ((A_2+A)x,x)\\ &\Rightarrow A_1+A \leq A_2+A\\ A_1 \leq A_2 &\Rightarrow (A_1x,x) \leq (A_2x,x) \qquad (2)\\ &\Rightarrow \alpha\,(A_1x,x) \leq \alpha\,(A_2x,x)\\ &\Rightarrow (\alpha A_1x,x) \leq \alpha\,(A_2x,x)\\ &\Rightarrow ((\alpha A_1)x,x) \leq ((\alpha A_2)x,x)\\ &\Rightarrow \alpha A_1 \leq \alpha A_2 \forall \alpha \geq 0. \end{aligned}$$

Hence theorem.

Positive Operator

Definition 9.15. A self-adjoint operator A is said to be **positive** if $A \geq 0$, i.e. $(Ax,x) \geq 0$ for all x.

It is clear that 0 and I are positive, as are T^*T and TT^* for an arbitrary operator T.

Theorem 9.16. If A is a positive operator on H, then $I+A$ is non-singular. In particular $I+T^*T$ and $I+TT^*$ are non-singular for an arbitrary operator T on H.

Proof. We must show that $I+A$ is one to one onto as a mapping of H into itself. First of all we observe that

$$\begin{aligned} &(I+A)(x) \Rightarrow x+Ax=0\\ &\Rightarrow Ax=-x \Rightarrow (Ax,x)=(-x,x) \geq 0.\\ &\Rightarrow -\|x\|^2 \geq 0 \Rightarrow x=0,\ \forall x \in H. \end{aligned}$$

Then

$$\begin{aligned} &(I+A)(x)=(I+A)y \Rightarrow (I+A)(x-y)=0.\\ &\Rightarrow x-y=0 \Rightarrow x=y\\ &\Rightarrow I+A \text{ is one-to-one.} \end{aligned}$$

It remains to show that $I+A$ is onto. It is sufficient to prove that range of $I+A$ equals H. Let M be the range of $I+A$. Then

$$\begin{aligned}
\|(I+A)x\|^2 &= \|x+Ax\|^2 = (x+Ax, x+Ax) \\
&= (x,x)+(x,Ax)+(Ax,x)+(Ax,Ax) \\
&= \|x\|^2 + 2(Ax,x) + \|Ax\|^2 \quad \text{[since } (Ax,x) \text{ is real]} \\
&\geq \|x\|^2 \\
\Rightarrow \qquad \|x\|^2 &\leq \|(I+A)x\|^2.
\end{aligned}$$

By this inequality and the completeness of H, M is complete and therefore closed. Suppose that $M \subset H$. Then a non-zero vector $x_0 \perp M$ such that

$$\begin{aligned}
& (x_0, (I+A)x_0) = 0 \\
\Rightarrow \quad & (x_0, x_0) + (x_0, Ax_0) = 0 \\
\Rightarrow \quad & \|x_0\|^2 + (Ax_0, x_0) = 0 \\
\Rightarrow \quad & \|x_0\|^2 + (Ax_0, x_0) \leq 0 \\
\Rightarrow \quad & x_0 = 0.
\end{aligned}$$

which contradicts the fact that x_0 is a non-zero vector.

Hence $M = H$. It follows therefore that $I+A$ is one-to-one and onto and hence non-singular.

Normal Operator

Definition 9.17. An operator N on a Hilbert space H is said to be **normal** if it commutes with its adjoint that is $NN^* = N^*N$.

Theorem 9.18. The set of all normal operators on H is a closed subset of $B(H)$ which contains the set of all self-adjoint operator and is closed under scalar multiplication.

Proof. If N is a self-adjoint operator, then

$$N^* = N \Rightarrow NN^* = N^*N.$$

Thus it follows that every self-adjoint operator is normal. Therefore the set M contains the set of all self-adjoint operators.

Let α be a scalar and N a normal operator, then

$$(\alpha N)(\alpha N)^* = (\alpha N)(\overline{\alpha} N^*)$$

$$
\begin{aligned}
&= \alpha\overline{\alpha}\,(NN^*) \\
&= \alpha\overline{\alpha}\,(N^*N) \\
&= (\overline{\alpha}N^*)\,(\alpha N) \\
&= (\alpha N)^*\,(\alpha N)
\end{aligned}
$$

Now consider the set M of all normal operators on H. It is clearly a subset of $\beta\,(H)$. To show that it is closed, it is sufficient to prove that every Cauchy sequence $\{N_k\}$ of normal operators on H converges to a normal operator. Due to the completeness of $\beta\,(H)$ this sequence converges to some operator N we shall show that N is normal. Since $N_k^* \to N^*$, we have

$$
\begin{aligned}
\|NN^* - N^*N\| &= \|NN^* - N_kN_k^* + N_kN_k^* - N_k^*N_k + N_k^*N_k - N^*N\| \\
&\leq \|NN^* - N_kN_k^*\| + \|N_kN_k^* - N_k^*N_k\ \| + \|N_k^*N_k - N^*N\| \\
&= \|NN^* - N_kN^*\| + \|N_k^*N_k - N^*\| \to 0 \\
&\leq \|NN^* - N_kN^*\| + \|N_k^*N_k - N^*\| \to 0
\end{aligned}
$$

which implies that

$$
\begin{aligned}
& NN^* - N^*N = 0 \\
\Rightarrow \quad & NN^* - N^*N
\end{aligned}
$$

therefore N is normal.

Theorem 9.19. If N_1 and N_2 are normal operators on a Hilbert space H with the property that either commutes with the adjoint of the other, then $N_1 + N_2$ and $N_1\ N_2$ are normal.

Proof. We are given that

$$
\begin{aligned}
& N_1N_1^* = N_1^*N_1, N_2N_2^* = N_2^*N_2 \\
& N_1N_2^* = N_2^*N_1, N_2N_1^* = N_1^*N_2
\end{aligned}
$$

We show first that $N_1 + N_2$ is normal. For this, we have.

$$
\begin{aligned}
(N_1 + N_2)\,(N_1 + N_2)^* &= (N_1 + N_2)\,(N_1^* + N_2^*) \\
&= N_1N_1^* + N_1N_2^* + N_2N_2^* + N_2N_2^* \\
&= N_1^*N_1 + N_2^*N_1 + N_1^*N_2 + N_2^*N_2
\end{aligned}
$$

$$= (N_1^* + N_2^*)(N_1 + N_2)$$
$$= (N_1 + N_2)^*(N_1 + N_2)$$

which shows that $N_1 + N_2$ is normal.

Similarly

$$\begin{aligned}(N_1N_2)(N_1N_2)^* &= (N_1N_2)(N_2^*N_1^*)\\ &= N_1(N_2N_2^*)N_1^*\\ &= N_1(N_2^*N_2)N_1^*\\ &= (N_1N_2^*)(N_1N_2^*)\\ &= (N_2^*N_1)(N_1^*N_2)\\ &= N_2^*(N_1N_1^*)N_2\\ &= (N_2^*(N_1^*N_1)N_2\\ &= (N_2^*N_1^*)(N_1N_2)\\ &= (N_1N_2)^*(N_1N_2)\end{aligned}$$

$\Rightarrow N_1N_2$ is normal.

Theorem 9.20. An operator on a Hilbert space H is **normal** if and only if

$$\|T^*x\| = \|Tx\| \text{ for every} x.$$

Proof. T is normal iff

$$\begin{aligned} & TT^* = T^*T \Leftrightarrow TT^* - T^*T = 0\\ \Rightarrow\quad & ((TT^* - T^*T)x, x) = 0 \quad \forall\, x \in H\\ & \qquad [\text{since an operator } T \text{ on } H \text{ is zero iff } (Tx,x) = 0\ ]\\ \Leftrightarrow\quad & (TT^*x, x) = (T^*Tx, x)\ \forall x \in H\\ \Leftrightarrow\quad & (T^*x, T^*x) = (Tx, Tx)\ \forall x \in H\\ \Leftrightarrow\quad & \|T^*x\|^2 = \|Tx\|^2\ \forall x \in H\\ \Leftrightarrow\quad & \|T^*x\| = \|Tx\|\ \forall x \in H\end{aligned}$$

Theorem 9.21. If N is a normal operator on H, then

$$\left\|N^2\right\| = \|N\|^2$$

Proof. Since N is normal, we have

$$
\begin{aligned}
& \|N^*x\| = \|Nx\| \quad \forall\, x \in H && (*)\\
\Rightarrow\quad & \left\|N^2\right\| = \sup\left\{\left\|N^2x\right\| ; \|x\| \le 1\right\} \\
& \qquad = \sup\{\|N(Nx)\| ; \|x\| \le 1\} \\
& \qquad = \sup\{\|N^*(Nx)\| ; \|x\| \le 1\} \\
& \qquad = \sup\{\|N^*Nx\| ; \|x\| \le 1\}
\end{aligned}
$$

[by the property of adjoint operation on $\beta\,(H)$]

Remark 9.22. For an arbitrary operator T on a Hilbert space, we form

$$A_1 = \frac{T+T^*}{2}, \qquad A_2 = \frac{T-T^*}{2}$$

It can be shown that A_1 and A_2 are self adjoint and they have the property that

$$T = A_1 + iA_2$$

In fact

$$
\begin{aligned}
A_1^* = \frac{1}{2}(T+T^*)^* &= \frac{1}{2}(T^*+T) \\
&= \frac{T+T^*}{2} = A_1 \Rightarrow A_1 \text{ is self-adjoint}
\end{aligned}
$$

and

$$
\begin{aligned}
A_2^* &= \left[\frac{1}{2i}(T-T^*)\right]^* = -\frac{1}{2i}(T-T^*) \\
&= \frac{1}{2i}(T-T^*) = A_2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad & A_2 \text{ is self-adjoint.} \\
& A_1 + iA_2 = \frac{T+T^*}{2} + \frac{T-T^*}{2} = T
\end{aligned}
$$

Theorem 9.23. If T is an operator on H, then T is normal $\Leftrightarrow$ its real and imaginary parts commute.

Proof. If A_1 and A_2 are real and imaginary parts of T so that $T = A_1 + iA_2$ and $T^* = A_1 - iA_2$,then

$$TT^* = (A_1+iA_2)(A_1-iA_2) = A_1^2 + A_2^2 + i(A_2A_1 - A_1 - A_2)$$

and

$$T^*T = (A_1 - iA_2)(A_1 + iA_2) = A_1^2 + A_2^2 + i(A_1A_2 - A_2A_1)$$

It is clear that if $A_1A_2 = A_2A_1$. Then $TT^* = T^*T$.

Conversely T is normal iff

$$\begin{aligned} & TT^* = T^*T \\ & \qquad \Leftrightarrow \quad A_1A_2 - A_2A_1 = A_2A_1 - A_1A_2 \\ & \qquad \Leftrightarrow \quad 2A_1A_2 = 2A_2A_1 \\ & \qquad \Leftrightarrow \quad A_1A_2 = A_2A_1. \end{aligned}$$

Unitary Operator

Definition 9.24. An operator U on H is said to be **unitary** if $UU^* = U^*U = I$.

Theorem 9.25. If T is an operator on H, then the following conditions are all equivalent to one another.

(1) $T^*T = I$

(2) $(Tx, Ty) = (x, y)$ for all x and y

(3) $\|T(x)\| = \|x\|$ for all x.

Proof. (1)$\Rightarrow$(2).

If $T^*T = I$, then

$$(Tx, Ty) = (x, T^*Ty) = (x, Iy) = (x, y)$$

for all x and y

(2)$\Rightarrow$(3). If $(Tx, Ty) = (x, y) = \|x\|^2$ for all x and y, then taking $y = x$, we have

$$\begin{aligned} & (Tx, Tx) = (x, x) = \|x\|^2 \\ \Rightarrow \quad & \|(Tx)\|^2 = \|x\|^2 \\ \Rightarrow \quad & \|(Tx)\| = \|x\| \quad \forall\, x. \end{aligned}$$

(3)$\Rightarrow$(1) when

$$\begin{aligned} & \|T(x)\| = \|x\| \\ \Rightarrow \quad & \|T(x)\|^2 = \|x\|^2 \end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \quad (Tx, Tx) = (x, x)\\
&\Rightarrow \quad (T^*Tx, x) = (Ix, x)\\
&\Rightarrow \quad ((T^*T - I)x, x) = 0 \quad \forall\, x \in M\\
&\Rightarrow \quad T^*T - I = 0\\
&\Rightarrow \quad T^*T = I.
\end{aligned}
$$

Theorem 9.26. An operator T on H is **unitary** iff it is an isometric isomorphism of H onto itself.

Proof. If T is unitary, then we know from the definition that it is onto.

Moreover since $T^*T = I$, by the previous Theorem.

$$\|T(x)\| = \|x\| \quad \forall\, x \in H.$$

Hence T is an isometric isomorphism of H onto itself.

Conversely if T is an isometric isomorphism of H onto itself, then T is a one-one mapping onto H such that

$$\|T(x)\| = \|x\| \quad \forall\, x \in H$$

and so by the above theorem, $T^*T = I$.

Since T is an isometric isomorphism of H onto itself, T^{-1} exists and then

$$T^*T = I \Rightarrow T^* = T^{-1}.$$

Also we note that

$$
\begin{aligned}
& \quad TT^* = TT^{-1} = I\\
&\Rightarrow \quad T^*T = TT^* = I\\
&\Rightarrow \quad T \text{ is unitary.}
\end{aligned}
$$

CHAPTER 10

Projections and Orthonormal Sets in Hilbert Spaces

We know that a projection on a Banach space B is an idempotent operator on B i.e. an operator P with the property $P^2 = P$. It was proved that each projection P determines a pair of closed linear subspaces M and N, the range and null space of P such that $B = M \oplus N$ and also conversely that each such pair of closed linear subspaces M and N determines a projection P with range M and null space N.

The structure which a Hilbert space H enjoy in addition to being a Banach space enables to single out for special attentions those projections whose range and null space are orthogonal.

We start the chapter with following theorem:

Theorem 10.1. If P is a projection on H with range M and null space N, then $M \perp N \Leftrightarrow P$ is self-adjoint and in this case $N = M^{\perp}$.

Proof. Since P is projection on a Hilbert space H with range M and null space N, we have $H = M \oplus N$, so each vector $z \in H$ can be written uniquely in the form $z = x + y$, $x \in M$, $y \in N$.

If $M \perp N$, then $(x,y) = (y,x) = 0$. Therefore for all z in H, we have

$$\begin{aligned}(P^*z,z) &= (z,Pz) = (z,x) = (x+y,x)\\ &= (x,x) + (y,x) = (x,x).\end{aligned}$$

and

$$\begin{aligned}&(Pz,z) = (x,z) = (x,x+y) = (x,x) = (x,y) = (x,x)\\ \Rightarrow\quad &(P^*z,z) = (Pz,z)\\ \Rightarrow\quad &[(P^*-P)z,z] = 0\\ \Rightarrow\quad &P^* - P = 0 \Rightarrow P^* = P.\end{aligned}$$

Conversely suppose that $P^* = P$, to prove that $M \perp N$, it is sufficient to show that if x and y are arbitrary elements of M and N respectively, then $(x,y) = 0$.

In fact we have,

$$(x,y) = (Px,y) = (x,P^*y) = (x,Py)$$

$$= (x,0) = 0. \quad \{N \text{ is the null space } y \in N, P(y) = 0\}$$

Hence $M \perp N$.

It remains to prove that if $M \perp N$. Then $N = M^{\perp}$. It is clear that $N \subseteq M^{\perp}$ and if N is a proper subset of $M^{\perp}$ and therefore a proper closed linear subspace of the Hilbert space $M^{\perp}$, there exists a non-zero vector z_0 in $M^{\perp}$ such that $z_0 \perp N$.

Since $z_0 \perp M$ and $z_0 \perp N$ and $H = M \oplus N$. It follows that $z_0 \perp H$. This is impossible and hence $N = M^{\perp}$.

Definition 10.2. A projection on H whose range and null space are orthogonal is called a prependicular projection.

The only projections considered in the theory of Hilbert spaces are those which are perpendicular.

In the light of above theory an operator P on a Hilbert space H is a perpendicular projection if $P^2 = P$ and $P^* = P$.

Moreover P is projection on M only if $(I - f)$ is a projection on $M^{\perp}$.

Theorem 10.3. If P and Q are the projections on closed linear subspaces M and N of H. Then $M \perp N \Leftrightarrow PQ = 0 \Leftrightarrow QP = 0$.

Proof. If $M \perp N$, then $N \subseteq M^{\perp}$. Since Q is a projection on N, Qz is in N for each $z \in H$.

Therefore

$$Qz \in M^{\perp} \Rightarrow P(Qz) = 0 \qquad\qquad \Rightarrow PQ(z) = 0 \Rightarrow PQ = 0.$$

Moreover taking adjoint, we have

$$PQ = 0 \Rightarrow (PQ)^* = 0^*$$

Hence $M \perp N \Rightarrow PQ = 0QP = 0$.

Conversely suppose that $QP = 0$

$$\begin{aligned}
&\Rightarrow PQ = 0, \text{ then for } x \in M \text{ or } y \in N,\\
&\qquad = (x,y) = (Px, Qy) = (x, P^*Qy)\\
&\qquad = (x, PQy) = (x, 0.y) = (x,0) = 0.
\end{aligned}$$

Hence $M \perp N$.

Therefore $QP = 0 \Rightarrow PQ = 0 \Rightarrow M \perp N$.

Definition 10.4. Two projections P and Q are orthogonal if $PQ = 0$.

Theorem 10.5. If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspaces $M_1, M_2, \ldots, M_n$ of H, then $P = P_1 + P_2 + \ldots + P_n$ is a projection $\Leftrightarrow$ $P_i's$ are pairwise orthogonal (in the sense that $P_iP_j = 0$ whenever $i \neq j$) and in this case, P is the projection on $M = M_1 + M_2 + \ldots + M_n$.

Proof. Each P_i is a perpendicular projection therefore $P_i^* = P_i = P_i^2$ for $i = 1, 2, \ldots, n$. Then

$$\begin{aligned} P^* &= (P_1 + P_1 + \ldots + P_n)^* = P_1^* + \ldots + P_1^* \\ &= P_1 + P_2 + \ldots + P_1 + = P. \end{aligned}$$

Hence P is self-adjoint. Now P is a projection $i \neq j$ it is idempotent.

If P_i's are pairwise orthogonal, then

$$P_iP_j = 0 \text{ for } i \neq j$$

Hence

$$\begin{aligned} P^2 &= (P_1P_2 + \ldots + P_n)^2 \\ &= \sum_{i=1}^{n} P_i^2 + 2\sum_{i \neq 1} P_iP_j \\ &= \sum_{i=1}^{n} X = P_i \qquad [\because\ P_i^2 = P_i \text{ and } PiPj = 0] \\ &= P \end{aligned}$$

$$\Rightarrow \quad P \text{ is idempotent.}$$

Thus we have proved that if P are pairwise orthogonal, then P is a projection.

To prove the converse we assume that P is idempotent. Let x be a vector in the range of P_i so that $P_i(x) = x$.

Then

$$\begin{aligned} \|x\|^1 &= \|P_i(x)\|^2 \\ &\leq \sum_{j=1}^{n} \left\|P_j(x)\right\|^2 \\ &= \sum_{j=1}^{n} \left(P_jx, P_j(x)\right) \sum_{j=1}^{n} \left(P_jx, P_j^*x\right) \end{aligned}$$

$$= \sum_{j=1}^{n} \left(P_j^2 x, x\right)$$
$$= \sum_{j=1}^{n} \left(P_j x, x\right)$$
$$= [(P_1 + P_2 + \ldots + P_n)x, x]$$
$$= (Px, x) = \left(P^2 x.x\right)$$
$$= (Px, P^* x)$$
$$= (Px, Px) \, \|Px\|^2 \le \|x\|^2$$

Since

$$\|x\|^2 = \|Px + (I - P)x\|^2$$
$$= \|Px\|^2 + \|(I + P)x\|^2 \quad \text{[Pythagorean theorem]}$$
$$\Rightarrow \qquad \|P(x)\|^2 \le \|x\|^2$$

Hence

$$\|x\|^2 \le \sum_{j=1}^{n} \left\|P_j(x)\right\|^2 \le \|x\|^2 \Rightarrow \|x\|^2 = \sum_{j=1}^{n} \left\|P_j(x)\right\|^2$$
$$\sum_{j=1}^{n} \left\|P_j x\right\|^2 = \left\|P_j(x)\right\|^2 = \|x\|^2$$

[Since $\|P_i(x)\|^2 = \|x\|^2$].

which implies that $\left\|P_j(x)\right\| = 0$ for $j \ne i$.

Now $P_i(x) = 0 \Rightarrow x \in$ Null space of P_j for $j \ne i$. Thus range of P_i is contained in the null space of P_i i.e. $M_i \subseteq M_j^{\perp}$ for every $i \ne j$ and this means that $M_i \perp M_j$ for $i \ne j$. Hence [by the preceding theorem] P_i's are pairwise orthogonal.

We now show that P is a projection on M. Firstly we observe that since $\|P(x)\| = \|x\| \, \forall x \in M_i$,, each M_i is contained in the range of P and therefore $M = \sum_{i=1}^{n} M_i$ is also contained in the range of P.

Secondly if x is a vector in the range of P, then

$$x = Px = (P_1 + P_2 + \ldots + P_n)x.$$
$$= P_1 x + P_2 x + \ldots. + P_n x$$

is evidently in $M = \sum_{i=1}^{m} M_i$ since $P_i x \in M_i$.

Hence the theorem.

Definition 10.6. A closed linear subspace M of a Hilbert space H is said to be invariant under an operator T on H if $T(M) \subseteq M$.

If both M and $M^{\perp}$ are invariant under T, then we say that M reduces T (or that T is reduced by M).

Theorem 10.7. A closed linear subspace M of H is invariant under an operator $T \Leftrightarrow M^{\perp}$ is invariant under T^*.

Proof. Suppose first that M is invariant under an operator T, then $T(x) \in M$ for all $x \in M$. We shall show that $M^{\perp}$ is invariant under T^*. If y is any vector of $M^{\perp}$.

Then

$$(x,y) = 0 \text{ for all } x \in M.$$
$$(x,T^*y) = (Tx,y) = 0 \text{ since } Tx \in M.$$
$$\Rightarrow \quad T^*y \in M^{\perp} \text{ for all } yy \in M^{\perp}$$

Hence $M^{\perp}$ is invariant under T^*.

Conversely suppose that M is invariant under T^*. Then M is invariant under $(T^*)^* = T^{**}$. But $M^{\perp\perp} = M$ and $T^{**} = T$.

Therefore it follows that M is invariant under T.

Theorem 10.8. A closed linear subspace M of H reduces an operator $T \Leftrightarrow M$ is invariant under both T and T^*.

Proof. By definition we know that M reduces T

$\Leftrightarrow M$ is invariant under T and $M^{\perp}$ is invariant under T

$\Leftrightarrow M$ is invariant under T and M is invariant under T^* [By previous Theorem].

$\Leftrightarrow M$ is invariant under both T and T^*.

Theorem 10.9. If P is a projection on a closed linear subspace M of H, then M is invariant under an operator $T \Leftrightarrow TP = PTP$.

Proof. If M is invariant under T and x is an arbitrary vector in H, then

$$x \in H \Rightarrow P(x) \in M \Rightarrow T(P(x)) \subset M$$
$$\Rightarrow \quad TP(x) \in M$$
$$\Rightarrow \quad P(TP(x)) = TP(x)$$

$$\Rightarrow \quad (\{PTP)(x) = TP(x)$$
$$\Rightarrow \quad PTP = TP.$$

Conversely if $TP = PTP$ and x is a vector in M then

$$P(x) = x$$
$$\Rightarrow \quad T(P(x)) = T(x)$$
$$\Rightarrow \quad PT(P(x)) = T(x)$$

But $TP(x) \in M$, therefore $T(x) \in M$.

Hence M is invariant under T.

Theorem 10.10. If P is the projection on a closed linear subspaces M of H, then M reduces an operator $T \Leftrightarrow TP = PT$.

Proof. By a result proved above, M reduces T iff M is invariant under T and

T^* iff $TP = PTP$ and $T^*P = PT^*P$

$\Leftrightarrow TP = PTP$ and $(T^*P)^* = (PT^*P)^*$

$\Leftrightarrow TP = PTP$ and

$P^*T^{**} = P^*T^{**}P^* \Leftrightarrow TP = PTP$

and $PT = PTP$ [$\because$ '$P^* = P$ and $T^{**} = T$]

$\Leftrightarrow TP = PT.$

Reflexivity of Hilbert space

Let H be a Hilbert space with inner product denoted by (y,x). The dual (conjugate space) H^* is then a Hilbert space with inner product given by $(x^*,y^*) = (y,x)$ for each x^* and y^* in H^* where $x \to x^*$ and $y \to y^*$ under the mapping $H \to H^*$.

We now establish the following result concerning the reflexivity of a Hilbert space.

Theorem 10.11. Every Hilbert space is reflexive.

Proof. Let H^* denote the dual space of a Hilbert space H. Consider the mapping T defined by

$$T : H \to H^*$$
$$y \to Ty = f \tag{1}$$

where the bounded linear functional f is, for any $x \in X$, given by

$$(Ty)(x) = f(x) = (x, y) \tag{2}$$

Suppose now that under T,

$$y_1 \to f_1$$

and

$$y_2 \to f_2$$

and let $y_1 \to f_1 \to g$

Thus

$$\begin{aligned} g(x) &= (x, y_1 + y_2) \\ &= (x, y_1) + (x, y_2) \\ &= f_1(x) + f_2(x) \end{aligned}$$

and we conclude that

$$T(y_1 + y_2) = T(y_1) + T(y_2)$$

Showing that T is additive. Now suppose under $T, y \to f$. and for a scalar α, let $T(\alpha y) = h$, then

$$y(x) = (x, \alpha y) = \overline{a}(x, y) = \overline{a} f(x)$$

therefore

$$T(\alpha y) = \overline{a} T(y)$$

Showing that T is conjugate linear. Also, by Riesz-Representation theorem for bounded linear functionals on a Hilbert space, to each bounded linear function f, there exists a unique y, $y \in H$ such that for every $x \in H, f(x) = (x, y)$ and $\|f\| = \|y\|$. In view of this the mapping T is onto and further

$$\|f\| = \|Ty\| = (y)(y \to Ty = f)$$

Therefore T is norm-preserving mapping or isometry. As we know that an isometry is always a $1-1$ mapping.

Thus we have, the mapping T constitutes a $1-1$ onto isometric, conjugate linear mapping from a Hilbert space onto conjugate space. Thus we see that Hilbert space and their conjugate spaces are indistinguishable metrically and almost indistinguishable algebraically. [Almost because T is conjugate linear]

Let x^* be a bounded linear functional on H and $x \in H$ Denote $x^*(x) = [x, x^*]$. Consider the mapping

$$\begin{aligned} J : H &\to H^* H^* \\ x &\to x^{**} \end{aligned}$$

where for defining equation for Jx we have for any $x \in H^*$

$$x^{**}(x^*) = [x^*, x^{**}] = [x^*, x] = x^*(x) \tag{3}$$

We now show that x^{**} is a bounded linear functional. Let $x^* \in X^*$, then

$$\begin{aligned} |x^{**}(x^*)| &= |x^*(x)| \le |x^*\| \ \|x\| \\ \Rightarrow \quad \|x^{**}\| &\le \|x\| \end{aligned} \tag{*}$$

Further if $x = 0$, then

$$0 \le \|x^{**}\| \le 0.$$

And consequently $\|x^{**}\| = \|x\| = 0$

If x is a non zero vector, then there must be some bounded linear functional x_0^* with norm 1 such that $x_0^*(x) = \|x\|$. But

$$\begin{aligned} \|x^{**}\| &= \sup_{\|x^*\|=1} |x^{**}(x^*)| \\ &= \sup_{\|x^*\|=1} |x^*(x)| \\ &\ge |x^*(x)| = \|x\| \end{aligned} \tag{**}$$

Thus $\|x^{**}\| = \|x\|$.

$\Rightarrow J$ is an isometry. Since isometry is always a $1-1$ mapping, it follows that J is an isomorphism. It remains to show that J is onto. To this end, let f be an element of H^{**}. We must find $z \in H$ such that $Jz = f$. For T defined in (1) consider the functional g defined by

$$\begin{aligned} g &: \to f \\ x &\to \overline{f(T(x))} \end{aligned}$$

For $x_1 x_2 \in H$, consider

$$\begin{aligned} g(x_1+x_2) &= \overline{f(T(x_1+x_2))} \\ &= \overline{f(Tx_1+Tx_2)} \\ &= \overline{f(T(x_1))}+\overline{f(T(x_2))} \\ &= g(x_1)+g(x_2) \Rightarrow \qquad g \text{ is additive.} \end{aligned} \tag{4}$$

Now let $x \in H$, $\alpha \in F$, then

$$\begin{aligned} g(\alpha x) &= \overline{f(T\alpha x)} \\ &= \overline{f(\overline{\alpha}T(x))} \\ &= \overline{\overline{\alpha}f(T(x))} \\ &= \alpha.g(x) \end{aligned}$$

Hence g is linear.

Further since T is an isometry, we have

$$\begin{aligned} |g(x)| &= \left|\overline{f(T(x))}\right| = |f(Tx)| \leq \|f\| \ \|Tx\| \\ &= \|f\| \ \|x\| \end{aligned}$$

Thus g is bounded.

By Riesz-Representation Theorem, $\exists\, z \in H$ such that for all $x \in H$,

$$g(x) = (x,z)$$

or

$$\begin{aligned} & \overline{f(Tx)} = (x,z) \\ \Rightarrow \quad & f(Tx) = (z,x) \end{aligned} \tag{5}$$

On the other hand by the definition of J and T (using (2) and (3)

$$(Jz)(Tx) = z^{**}(Tx) = Tx(z) = (z,x) \tag{6}$$

Thus (5) and (6) yield that Jz and f agree on every member of H^*. Hence they are same. This completes the proof.

Example 10.12. Show that a Hilbert space is finite dimensional $\Leftrightarrow$ every complete orthonormal set is a basis.

Solution. Let H be a finite dimensional Hilbert space of dimensional n. Let $S =< e_i >$ be a complete orthonormal set in H. Then we have to show that S is a basis for H. Since S is an orthonormal set, therefore it is linearly independent.

Also S must be a finite set because it can not contain more than n vectors. [since H is finite dimensional]. Now let $x \in H$. Since S is a complete orthonormal set, therefore we have $x = \sum_{e_i \in s} (x, e_i) e_i$. Thus each vector x in H can be written as linear combination of vectors in the set S and so S generates H. Therefore S is a basis for H. [Thus in a finite dimensional Hilbert space of dimension n every complete orthonormal set must contain exactly n vectors].

Conversely suppose that every complete orthonormal set in a Hilbert space H is a basis for H. Then to show that H is finite dimensional. Let S be a complete orthonormal set in H. Then by hypothesis S is a basis for H. We are to show that S is infinite set. Suppose is infinite. Then we can certainly extract a denumerable sequence of distinct points of S

$$e_1, e_2, e_3, \ldots, e_n, \ldots$$

Consider now the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} e_n.$$

Since the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is convergent $\Rightarrow$, the series $\sum_{n=1}^{\infty} \frac{1}{n^2} e_n$. is convergent [by the result that. Let H be a Hilbert space and let $S =< e_1, e_2, e_3, \ldots, e_n, \ldots >$ be countably infinite orthonormal set in H. Then a series of the form $\sum_{n=1}^{\infty} \alpha_n e_n$ is convergent iff $\sum_{n=1}^{\infty} |\alpha_n|^2 < \infty$.

Thus the series $\sum_{n=1}^{\infty} \frac{1}{n^2} e_n$. must converge to some vector x in H. Since S is a basis for H, therefore we can write x as some finite linear combination of vectors in S. Let

$$x = \alpha_\lambda e_\lambda + \ldots + \alpha_\mu e_\mu$$

where $e_\lambda, \ldots e_\mu \in S$ and, $\alpha_\lambda, \ldots, \alpha_\mu$, are scalars. Let j be any *positive* integer having value different from the values of indices $\lambda, \ldots, \mu$ We have

$$(x, e_j) = (\alpha_\lambda e_\lambda + \ldots + \alpha_\mu e_\mu e_j)$$
$$(x, e_j) = (\alpha_\lambda e_\lambda + \ldots + \alpha_\mu e_\mu, e_j) = 0.$$

Also

$$(x, e_j) = \left(\sum_{n=1}^{\infty} \frac{1}{n^2} e_n e_j. \right) \quad \left[\because\ x = \sum_{n=1}^{\infty} \frac{1}{n^2} e_n \right]$$

$$= \frac{1}{n^2}$$

Thus we have $= \frac{1}{n^2} = 0$ which is not possible. Therefore the set S must be finite and H is finite dimensional.

Theorem 10.13. Prove that any two complete orthonormal sets in a Hilbert space H have the same cardinal number.

Proof. Let S_1 and S_2 be two complete orthonormal sets in a Hilbert space H.

Suppose one of these sets is finite. Let S_1 be finite and $S_1 = \{e_1, e_2, \ldots, e_n\}$..

Since S_1 is an orthonormal set, therefore it is linearly independent. Also since S_1 is complete, therefore if $x \in H$, then we have

$$x = \sum_{i=1}^{n} (x, e_i)\, e_i$$

Thus S_1 generates H. Therefore S_1 is a basis for H and so H is finite dimensional and dim H = n. Since S_2 is also a complete orthonormal set in H, therefore S_2 must also be a basis for H. Since S1 and S_2 are both bases for H, therefore they must have the same number of elements.

Now let us suppose that both S_1 and S_2 are infinite sets. Let $x \in S_1$ and let $S_2(x) = \{y : y \in S_2 \text{ and } (y,x) \neq 0\}$. Then $S_2(x)$ is a subset of S_2 and thus $S_2(x)$ is a countable set . Let z be any arbitrary member of S_2. Since S_1 is a complete orthonormal set and therefore by Parseval's identity, we have

$$\|z\|^2 = \sum_{x \in S_1} |(z,x)|^2$$

But $z \in S_2 \Rightarrow z$ is a unit vector.

Therefore we have

$$1 = \sum_{x \in S_1} |(z,x)|^2.$$

From this relation we see that there must exist some vector $x \in S_1$ such that $(z,x) \neq 0$. Then by our definition of $S_2(x)$, we have $z \in S_2(x)$. Thus $z \in S_2 \Rightarrow z \in S_2(x)$ for some $x \in S_1$. Therefore we have

$$S_2 = \bigcup_{x \in S_1} S_2(x) \tag{1}$$

Let $n_1, n_{2,}$ be the cardinal numbers of S_1, S_2 respectively. Since the cardinal number of the union of an arbitrary collection of sets can not exceed the cardinal number of index set, therefore $n_2, \leq n_{1,}$. Interchanging the roles of S_1 and S_2 we get $n_1 \leq n_2$. Therefore we have $n_1 = n_2$.

Remark 10.14. Let S be a complete orthonormal set in a Hilbert space H. Then the cardinal number of S is said to be the orthogonal dimension of H. If H has no complete orthonormal set i.e. if H is the zero space, then the orthogonal dimensional of H is said to be zero.

Definition 10.15. Operators S and T are said to be metrically equivalent if $\|Sx\| = \|Tx\|$ $\forall\, x \in H$..

Theorem 10.16. Operators S and T are metrically equivalent if $S^*S = T^*T$

Proof. Let S and T be metrically equivalent

$$
\begin{aligned}
&\|Sx\| = \|Tx\| \quad \forall\, x \in H. \\
&\Leftrightarrow (S^*Sx, x) = (Sx, Sx) = \|Sx\|^2 = \|Tx\|^2 \\
&\qquad\qquad = (Tx, Tx) = (T^*Tx, x) \\
&\Rightarrow ((S^*S - T^*T)x, x) = 0 \\
&\Rightarrow S^*S - T^*T = 0 \\
&\Rightarrow S^*S = T^*T.
\end{aligned}
$$

Theorem 10.17. An operator T is normal iff T and T^* are metrically equivalent.

Proof. Suppose T is normal $\Rightarrow T^*T = TT^*$ and so by the above theorem, T^* and T are metrically equivalent.

Conversely suppose that T and T^* are metrically equivalent

$$
\begin{aligned}
\Rightarrow &\quad \|T^*x\| = \|Tx\| \\
\Rightarrow &\quad T^*T = TT^* \\
\Rightarrow &\quad T \text{ is normal.}
\end{aligned}
$$

Finite Dimensional Spectral Theory

First we give basic definitions and results.

Definition 10.18. Let T be an operator on a Hilbert space H. A vector $x \in H$ is said to be a proper vector (eigen-vector, latent vector or characteristic vector) for the operator T if (i) $x \neq 0$ and (ii) $Tx = ux$ for a suitable scalar u. if also $Tx = vx$, then $Tx = ux$ and $Tx = vx$ implies $(u-v)x = 0$. Since $x \neq 0$, it follows that $u = v$. Thus a proper vector x determines uniquely the associated scalar u.

Definition 10.19. A scalar u is said to be a proper value (Eigen value, latent root or characteristic root(value)) for the operator T in case there exists a non-zero vector x such that $Tx = ux$,.

Thus u is a proper value for T if and only if the null space of $(T - uI)$ is not equal to $\{0\}$.

Remark 10.20. If the Hilbert space H has no non-zero vector at all, then T certainly has no eigen vectors. In this case the whole theory collapses into triviality. So we assume throughout this lesson that $H \neq \{0\}$.

Theorem 10.21. If T is a normal operator, x is a vector and u is a scalar, then $Tx = ux$ if and only if $T^*x = ux$. In particular

(1) x is a proper vector for T if and only if it is a proper vector for T^*.

(2) u is a proper value of T if and only if it $\bar{u}$ is a proper value of T^*.

Proof. By virtue of normality, $T^*T = TT^*$.
Since

$$(T - uI)^* = T^* - \bar{u}I^* = T^* - \bar{u}I.$$

we have

$$(T - uI)^* = T^* - \bar{u}I^* = T^* - \bar{u}I.$$

and

$$(T - uI)^*(T - uI) = (T^* - \bar{u}I)(T - uI)$$

Since $TT^* = T^*T$, it follows that $T - uI$ is normal. Hence

$$\|(T - uI)x\| = \left\|(T - uI)^* x\right\|$$

which in turn implies that $Tx = ux$ if and only if $T^*x = \bar{u}x$. This proves (1) and (2).

Remark 10.22. Let H be a classical Hilbert space and $x_1, x_2, \ldots$ an orthonormal basis for H. Then one sided shift operator T defined by $Tx_k = x_{k+1}$ has no proper value.

Theorem 10.23. Let T be a normal operator on a Hilbert space H. Then there exists on orthonormal basis for H consisting of eigen vectors of T.

Proof. Let λ be an eigen value of T and suppose x is corresponding eigen vector. Thus we have $Tx = \lambda x$. Since x can not be zero, we can choose $x_1 = \frac{x}{\|x\|}$, If the dimension of H is 1, then we are done. If not, we will proceed by induction. We shall assume that the theorem is true for all spaces of dimension less than H and then show that it follows for x from this assumption.

Letting $m = [x_1] = \{\alpha x_1, \alpha \in F\}$. The space spanned by x_1, we have the following direct sum composition of H:

$$H = M \oplus M^{\perp}.$$

We must have then $\dim M^{\perp} < \dim H$. Since x_1 is an eigen vector of T, we have $Tx_1 = \lambda x_1$ and therefore it is clear that M is invariant under T. But we know by Theorem 1 that eigen vectors of T must also be eigen vectors for T^*. Therefore M is invariant under T^* also. Hence M is invariant under $T^{**} = T$. Thus we have

(i) M is invariant under T.
(ii) $M^{\perp}$ is invariant under T.

Thus we can say that M reduces T.

Consider now the restriction of T to $M^{\perp}$ denoted by $T/M^{\perp}$ where $T/M^{\perp} : M^{\perp} \to M^{\perp}$. Since T is normal, $T/M^{\perp}$ is also normal since $M^{\perp}$ reduces T. Now since $\dim M^{\perp} < \dim H$, we can apply the induction hypothesis to assert the existence of an orthonormal basis for M consisting of eigen vector for $T/M^{\perp}; \{x_1, x_2, \ldots, x_n\}$. Eigen vectors of $T/M^{\perp}$ however must also the eigen vector of T. Hence for the entire space, we have $(x_1, x_2, \ldots, x_n)$ as orthonormal basis of eigen vectors of T. Hence the result.

Spectral Theorem for Finite Dimensional spaces

Definition 10.24. The set of eigen values of an operator T is called its spectrum or point spectrum and is denoted by (T).

Statement of Spectral Theorem

Theorem 10.25. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigen values of an operator T and let $M_1, M_2, \ldots, M_n$ be their corresponding eigen spaces. If $P_1, P_2, \ldots, P_n$ are the projections on these eigen spaces, then the following three statements are equivalent to one another.

(1) $M_i^{'}s$ are pairwise orthogonal and span H.

(2) $P_i^{'}s$ are pairwise orthogonal, that is $P_iP_j = 0$ for $i \neq j$ and $I = P_1 + P_2 + \ldots + P_n$ and also

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n \lambda_n$$

(3) T is normal.

Proof. (1)$\Rightarrow$(2), by (1) every vector x in H can be expressed uniquely in the form

$$x = x_1 + x_2 + \ldots + x_n, \tag{4}$$

where $x_i \in M_i$ for each i and $x_i^{\prime}s$ are pairwise orthogonal. Further (1) $M_i \perp M_j, i \neq j$ then $M_j \subset M_i$. Then since $P_jx = M_j$ for every x, we have $P_iP_jx = 0$ for any x and $P_iP_j = 0$ for $i \neq j$. This proves that P_i's are pairwise orthogonal.

Applying P_i to both sides of (4), we have

$$\begin{aligned} P_ix &= P_ix_1 + P_ix_2 + \ldots + P_ix_n \\ &= 0 + 0 + \ldots + P_ix_i + \ldots + 0 \\ &= x_i \text{ for any } i. \end{aligned}$$

Hence we can write any x as

$$x = P_1x + P_2x + \ldots + P_nx$$

or $Ix = P_1x + P_2x + \ldots + P_nx$ for identity operator T.

or $Ix = (P_1 + P_2 + \ldots + P_nx)x$

Since this is true for any $x \in H$, we conclude that

$$I = P_1 + P_2 + \ldots + P_n$$

Further applying T to x in (4), we have

$$\begin{aligned}Tx &= Tx_1 + Tx_2 + \ldots + Tx_n\\ &= \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n\\ &= (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n)x\end{aligned}$$

for every x and so

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n \tag{5}$$

The representation (5) for an operator T, when it exists is called the Spectral Representation or Spectral form of T.
(2)⇒(3), it follows from

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n$$

That

$$\begin{aligned}T^* &= \overline{\lambda}_1 P_1^* + \overline{\lambda}_2 P_2^* + \ldots + \overline{\lambda}_n P_n^*\\ &= \overline{\lambda}_1 P_1 + \overline{\lambda}_2 P_2 + \ldots + \overline{\lambda}_n P_n\end{aligned}$$

Now since by (2) $P_i P_j = 0$ for $i \neq j$, we have

$$\begin{aligned}TT^* &= (\lambda_1 P_1 + \lambda_2 P_2 + \ldots + \lambda_n P_n)(\lambda_1 P_1 + \overline{\lambda}_2 P_2 + \ldots + \overline{\lambda}_n P_n)\\ &= |\lambda_1|^2 P_1^2 + \ldots + |\lambda_n|^2 P_n^2\\ &= |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_n|^2 P_n\end{aligned}$$

and similarly

$$T^*T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \ldots + |\lambda_n|^2 P_n$$

and therefore

$$TT^* = T^*T.$$

Proving that T is normal.

(3)⇒(1): Suppose that T is normal.

We shall prove first that $M_i \perp M_j$ for $i \neq j$. Given $x_i \in M_i, x_j \in M_i,$, it is sufficient to show that $x_i \perp x_i$. Since $x_i \in M_i, x_j \in M_i,$, we have $Tx_i = \lambda_i x_i, Tx_j = \lambda_j x_j$.

Since T is normal $Tx_i = \lambda_i x_i, Tx_j = \lambda_j x_j$ and so

$$\left(Tx_i, x_j\right) = \left(x_i, T^* x_j\right)$$

or $\left(\lambda_i x_i, x_j\right) = \left(x_i, \lambda_i x_j\right)$
or $\lambda_i \left(x_i, x_j\right) = \lambda_j \left(x_i, x_j\right)$
or $\left(\lambda_i, \lambda_j\right) \left(x_i, x_j\right) = 0$

Since $\lambda_i \neq \lambda_j$, it follows that $\left(x_i x_j\right) = 0$ and hence $x_i \perp x_j$. This proves that $M_i \perp M_j$ for i, j and so M_i's are pairwise orthogonal. It remains to prove that T is normal, then M_i's span H that is. We have just $H = M_1 + M_2 + \ldots + M_n$. Shown that $M_i' s$ are pairwise orthogonal. This implies that P_i's are pairwise orthogonal. Therefore $M = M_1 + M_2 + \ldots + M_n$ is a closed linear subspace of H and its associated projection is $P = P_1 + P_2 + \ldots + P_n$. Also we know that if T is normal, then M_i reduces T. Therefore $TP_i = P_i T$ for each P_i, it follows from this that $TP = PT$ and hence M reduces T and so by definition M is invariant under T. If $M \neq \{0\}$, then since all the eigen vectors of T are contained in M, the restriction of T to M is an operator (normal) on a non-trivial finite dimensional Hilbert space which has no eigen vectors and hence no eigen values. But this is a contradiction to the fact that there exists an orthonormal basis for H consisting of eigen vectors of normal operator T. Hence $M = \{0\}$ and so $M = H$ and hence which $H = M_1 + M_2 + \ldots + M_n$ shows that M_i's span H.

Hence the result.

Suggested Readings:

1. S. Ponnusamy, Foundation of Functional Analysis, Alpha Science International Ltd, 2002, ISBN 10: 1842650793 / ISBN 13: 9781842650790.

2. P.K. Jain and O.P. Ahuja, Functional Analysis, New Age International, 1995, ISBN 812240801X / 9788122408010.

3. D. Somasundaram, A First Course in Functional Analysis, Narosa Publishing House, 2014, ISBN-13: 9788173197437.

4. Walter Rudin, Functional Analysis , McGraw Hill Education, 2006, ISBN: 9780070619883.

5. Erwin Kreyszig, Introduction Functional Analysis with Application, John Wiley and Sons Ltd, 1978, ISBN 10: 047103729X / ISBN 13: 9780471037293.

6. Casper Goffman and Georege Pedrick, First Course in Functional Analysis,Chelsea Publishing, 1983, ISBN-100828403198 / ISBN-139780828403191.